22.1. The first best is characterized by the possibility that action can be observed and therefore contract can be made contingent on both action and performance. Making the contract contingent on performance brings some uncertainty, but it is important to note that the uncertainty is not due to the variation in action as actions are perfectly observed.

The agent (A), being risk averse, is concerned about the variation in wage payment. In particular, under our assumption of CARA risk preferences and normally distributed error, we know (recall our discussion of certainty equivalent) that A maximizes

$$Ew - \frac{\eta}{2} var(w) - \psi(a_0 + a_1 + \ldots + a_n) + v_1(a_1) + \ldots + v_n(a_n).$$

Principal (P), being risk neutral, maximizes $E\pi$.

$$\begin{aligned} E\pi &= E(q-w) \\ &= E(a_0 + \varepsilon - w) \\ &= a_0 - Ew. \end{aligned}$$

IR constraint : $Ew - \frac{\eta}{2} var(w) - \psi(a_0 + a_1 + \ldots + a_n) + v_1(a_1) + \ldots + v_n(a_n) \geq w_0$

Following our regular argument, one can easily show that IR constraint should be binding in equilibrium. So we can replace Ew in P's objective function and rewrite his maximization problem as

$$\max_{a,w} a_0 - w_0 - \frac{\eta}{2} var(w) - \psi(a_0 + a_1 + \ldots + a_n) + v_1(a_1) + \ldots + v_n(a_n) \tag{0.1}$$

Inspecting (0.1), it can be argued that P would prefer a fixed wage schedule, because any variation in w would only reduce the value of the objective function in (0.1). Hence we conclude that the optimal w must satisfy $var(w) = 0$. We can therefore write the first order condition of the above problem as follows:

$$\begin{aligned} FOC \ : \ & \psi'(a_0 + a_1 + \ldots + a_n) = 1 \\ & v_i'(a_i) = \psi'(a_0 + a_1 + \ldots + a_n) \end{aligned} \tag{0.2}$$

The optimal level of action, $a^* = (a_0^*, a_1^*, \ldots, a_n^*)$, satisfies (0.2). From the IR constraint, we see that the optimal first best wage is given by

$$w^* = w_0 + \psi(a_0^* + a_1^* + \ldots + a_n^*) - v_1(a_1^*) - \ldots - v_n(a_n^*).$$

22.2. Next consider the case with linear contract $t + sq \equiv t + sa_0 + s\varepsilon$ and assume that action cannot be observed. Further assume that P restricts possible actions that A can take, and the restrictive set (in equilibrium) be denoted by S (note that we will have to solve for S ultimately).

So we have to consider the following IC constraint

$$\underline{a} = \arg\max t + sa_0 - \frac{\eta}{2} s^2 \sigma^2 - \psi\left(a_0 + \sum_{i \in S} a_i\right) + \sum_{i \in S} v_i(a_i)$$

The above objective function is nothing but the certainty equivalent of the wage A gets (recall our discussion on linear contract with CARA utility function). From the first order condition of the agent's problem, once can easily see that the optimal actions must satisfy

$$
\begin{aligned}
(0.3) \qquad FOC \quad : \quad & \psi'\left(a_0 + \sum_{i \in S} a_i\right) = s \\
& v_i'(a_i) = \psi'\left(a_0 + \sum_{i \in S} a_i\right) \text{ for all } a_i, i \in S \\
\implies \quad & v_i'(a_i) = s
\end{aligned}
$$

Even before we solve the principal's problem, two comments are worth mentioning here. First, if s differs from 1, the equilibrium actions would be different from the first best actions. Second, the equilibrium time spent on any task only depends on the value s, but not on the set S.

The IR constraint will be given by

$$
t + sa_0 - \frac{\eta}{2} s^2 \sigma^2 - \psi\left(a_0 + \sum_{i \in S} a_i\right) + \sum_{i \in S} v_i(a_i) \geq w_0 \tag{0.4}
$$

Next, using our regular argument of binding IR (even if I am skipping this argument, you should not! Be explicit why we can argue that IR is binding), we can write the principal's objective function as

$$
\begin{aligned}
E\pi &= E(q - w) = E(q - t - sq) \\
&= E(a_0 + \varepsilon - t - s(a_0 + \varepsilon)) = a_0 - t - sa_0 \\
&= a_0 - w_0 - \frac{\eta}{2} s^2 \sigma^2 - \psi\left(a_0 + \sum_{i \in S} a_i\right) + \sum_{i \in S} v_i(a_i) \qquad (0.5)
\end{aligned}
$$

So given a set of tasks (permitted in equilibrium), S, the optimal time spent on each task should solve the first order condition of $\max E\pi$.

The first part in Q 22.2 asks you to find out S, given a value of s. In order to find that you do not have to explicitly solve for the optimal values of a_i; Instead, consider the effect of a_i on P's expected profit in (0.5). From (0.3), notice that total time spent by the agent, $a_0 + \sum_{i \in S} a_i$, and time spent on each task $i \in S$, depend only on s. Let us denote the equilibrium time spent on task i by $a_i(s)$. Therefore, if task k is dropped from set S, then total effort time and all a_is other than a_0 being fixed, time spent on task 0, a_0 is going to increase to $a_0 + a_k(s)$.

This will increase by $E\pi$ by $a_k(s)$ (look at the first term in (0.5)) and $v_k(a_k(s))$ will be reduced from $E\pi$ (look at the second term in (0.5)). P will therefore drop task k if $v_k(a_k(s)) - a_k(s)$ is negative. Or in other words, we can write, S consists of all such tasks satisfying $S(s) = \{k \leq n \mid v_k(a_k(s)) - a_k(s) > 0\}$.

Next, consider the possibility that $s = 1$. Easy to see that choice of optimal a_is will coincides with the first best allocation. Will all the tasks will be included in S? The answer is yes! Can you think about the reason? I will give you a hint: try to see where $v_i(a_i) - a_i$ is maximized!!

Next, assume $s < 1$. Since v_is are concave, optimal time spent on task i will fall. The set of optimal tasks will also shrink, as $v_i(a_i(s)) - a_i(s)$ will not necessarily be positive for all values of i.