Problem Set 3
Due 06. October, 14:15

Exercise 3.1: Overlapping generations and bubbles

Consider an endowment economy with overlapping generations where agents live for two periods. Agents have preferences over consumption when young, c_t^y, and old, c_t^o, and do not discount the future

$$U_t = u\left(c_t^y\right) + u\left(c_{t+1}^o\right), \quad u(c) = \frac{c^{1-\sigma} - 1}{1-\sigma}, \quad \sigma > 1,$$

Agents' endowment when young, ω^y, is higher then the one in the old age, $0 < \omega^o < \omega^y$. There are two assets in this economy. A one-period bond b_t with price 1 that yields a safe interest r_t per unit and is available in zero net supply (think of it as private lending), and a console bond (an infinitely-lived bond) denoted by a_t with price q_t which pays a fixed dividend $d \geq 0$ in every period and is available in unit supply. For simplicity, also assume that the size of each cohort of agents is equal to one.

(a) Derive the equations that characterize the competitive equilibrium of this economy.

(b) Show that the price of the console bond satisfies the pricing rule

$$q_t = \frac{q_{t+1} + d}{1 + r_{t+1}}. \tag{1}$$

(c) Let a **rational asset bubble** be present when an asset's value is larger than the present value of its dividends. Assuming that $d = 0$, show that there exist two stationary equilibria (equilibria where the interest rate, $r_{t+1} > -1$, remains constant over time) - one with a rational asset bubble and one without. What is the equilibrium interest rate in each of these stationary equilibria?

(d) Suppose the economy is in a stationary equilibrium with a rational asset bubble (the price of the console bond is strictly positive). What happens to the price of the console bond and the consumption of the current young and old if the beliefs of the young agents shift suddenly so they expect the bond will have zero value the next period?

(e) Based on your answer above, would you say the bubble equilibrium is Pareto superior to the equilibrium without the bubble?

(f) Finally, consider now an economy with strictly positive dividends from the console bond, $d > 0$. Compute the forward solution of the console's price, q_t. How many stationary equilibria with a finite console price are there in this economy? Characterize the stationary equilibrium interest rate.

Exercise 3.2: Value function iteration with paper and pencil

Consider the following version of the neoclassical growth model in discrete time

$$V(k) = \max_{c \geq 0, k' \geq 0} \log(c) + \beta V(k') \text{ s.t. } c + k' \leq k^\alpha + (1-\delta)k,$$

where $V(k)$ is the value function, c denotes consumption, k the physical capital stock, δ is the depreciation rate of capital, and α is the capital income share of the economy. Variables with primes denote on period forward variables (for example, if $k = k_t$ then $k' = k_{t+1}$), this notation reflects that fact that the Bellman equation is independent of calendar time.

(a) Assume that depreciation is 100% and that the resource constraint holds with equality. Start with the guess that the value function is $V^0(k) = 0$. Update your guess by iterating on the the Bellman equation

$$V^{j+1}(k) = \max_{0 \leq k' \leq k^\alpha} \log(k^\alpha - k') + \beta V^j(k'), \quad j = 0, 1, \ldots,$$

and show that the value function converges to $V^{j+1}(k) = A + B\log(k)$ as $j \to \infty$. Pin down the value of B.

(b) Use the stationary Bellman equation

$$V(k) = \max_{k' \geq 0} \log(k^\alpha - k') + \beta \left[A + B\log(k')\right],$$

to show that

$$g(k) = \alpha\beta k^\alpha$$
$$A = \frac{1}{1-\beta}\left[\log(1-\alpha\beta) + \frac{\alpha\beta}{1-\alpha\beta}\log(\alpha\beta)\right],$$

where

$$g(k) = \arg\max_{k' \geq 0} \log(k^\alpha - k') + \beta \left[A + B\log(k')\right].$$

Exercise 3.3: Solving the Neoclassical Growth Model with Function Approximation

Consider the following version of the neoclassical growth model in discrete time

$$V(k) = \max_{c \geq 0, k' \geq 0} \frac{c^{1-\sigma} - 1}{1-\sigma} + \beta V(k') \text{ s.t. } c + k' \leq k^\alpha + (1-\delta)k,$$

where $\alpha = .36$, $\beta = .95$, $\delta = .1$, $\sigma = 2$. As preferences are not logarithmic and depreciation is less than 100%, there is no known closed form solution for the value function and the policy functions, so we will solve for the equilibrium with numerical approximation.

(a) Show that the following system of functional equations in the unknown functions $V(k)$, $k' = g(k)$, and $c(k)$ characterizes the dynamic equilibrium

$$\begin{aligned} V(k) &= u(c(k)) + \beta V(g(k)) \\ u'(c(k)) &= \beta \left[f'(g(k)) + (1-\delta) \right] u'(c(g(k))) \\ c(k) &= f(k) + (1-\delta)k - g(k), \end{aligned}$$

where k is the state variable and $u(c)$ and $f(k)$ denote the known utility and the production function, respectively.

(b) Compute the steady-state capital stock, $k^\star = g(k^\star)$, as well as the associated consumption, $c(k^\star)$, and value, $V(k^\star)$ as a function of the model's parameters only.

(c) Read the following sections of Miranda and Fackler (2002)

- Appendix B: A Matlab Primer.
- Sections 6.1, 6.2, 6.6, and 6.7 on Function Approximation.

Then download the CompEcon Toolbox from the book's website:

http://www4.ncsu.edu/~pfackler/compecon/toolbox.html

and unzip the folder and save it in a directory where your Matlab can access it. I am using

M:/pc/Dokumenter/MATLAB/compecon2011_64/CEtools

for example, such that it also works with the Matlab provided by the UiO Programkiosk. Finally, choose in Matlab's Home tab "set path" and add the directories

- M:/pc/Dokumenter/MATLAB/compecon2011_64/CEtools
- M:/pc/Dokumenter/MATLAB/compecon2011_64/CEdemos

to Matlab's search path. Type **help cetools** in Matlab's command window to check whether you followed the steps correctly (the command should return a general description of the CompEcon Toolbox). Finally, click on "New" in Matlab's home path to create a new script and you are ready to go.

(d) Write a Matlab program that solves for the dynamic equilibrium (i.e., the above system of functional equations) using CompEcon's Function Approximation Toolkit described in Miranda and Fackler (2002, Section 6.6). **Send me your Matlab scripts per email before the due date**.