Problem Set 5
Due 27. October, 14:15

Exercise 5.1: Stochastic dynamic programming

Consider a stochastic version of the neoclassical growth model

$$V(k,z) = \sup_{0 \le k' \le \bar{k}} u\left(zk^\alpha + (1-\delta)k - k'\right) + \beta \sum_{z' \in Z} \pi(z'|z) V(k',z'),$$

where Z is the finite set of all possible realisations for the stochastic Markov productivity $z > 0$, and $\pi(z'|z)$ denotes the probability of transiting from state z today to state z' tomorrow. The upper bound $\bar{k}$ is chosen so high that it is never binding, but it makes the state space of k bounded. Let the utility function $u(\cdot)$ be twice differentiable with $u'(\cdot) > 0$, and $u''(\cdot) < 0$, and let it also satisfy the Inada conditions.

(a) Show that the operator on the right-hand side of the Bellman equation

$$T(V)(k,z) = \sup_{0 \le k' \le \bar{k}} u\left(zk^\alpha + (1-\delta)k - k'\right) + \beta \sum_{z' \in Z} \pi(z'|z) V(k',z')$$

is a contraction mapping. (Hint: check Blackwell's sufficient conditions.)

(b) Show that the value function $V(k,z)$ is strictly concave in k. (Hint: you cannot use calculus because equilibrium functions might not be differentiable. Show instead that T maps *concave* functions into *strictly concave* functions.)

(c) Show that the value function is differentiable in k with derivative

$$\frac{\partial V(k,z)}{\partial k} = u'(c(k,z))[z\alpha k^{\alpha-1} + (1-\delta)].$$

(Hint: use Benveniste and Scheinkman (1979, Lemma 1).)

(d) Show that

$$g(k,z) = \arg\max_{0 \le k' \le \bar{k}} u\left(zk^\alpha + (1-\delta)k - k'\right) + \beta \sum_{z' \in Z} \pi(z'|z) V(k',z'),$$

exists and is unique. (Hint: use the extreme value theorem for the existence.)

Exercise 5.2: An RBC model with indivisible labor

In the data, labor input is more volatile over the business cycle compared to labor productivity (compare the standard deviation of hours worked relative to the standard deviation of output per hour worked in Hansen (1985, Table 1, Column 1)[1]. The baseline real business cycle model with divisible labor is not able to replicate this fact (see Hansen 1985, Table 1, Column 3). By introducing the concept of indivisible labor, Hansen (1985) was able to address this puzzle.

[1]Hansen, G., "Indivisible Labor and the Business Cycle," Journal of Monetary Economics, 16 (1985), 281-308.

(a) Replicate Hansen (1985, Table 1, Columns 1-2) for the Norwegian mainland economy and check whether the above mentioned business cycle fact is also present in the Norwegian mainland economy. It is enough to compute the moments for output (GDP mainland, $x_{:,10}$), hours (Total hours worked mainland, $x_{:,12}$), and productivity (GDP mainland divided by total hours worked mainland).

Use the standard procedure: (i) normalize the time-series, (ii) HP-filter the time-series with a smoothing parameter $\lambda = 1600$, and (iii) compute standard deviations and cross-correlation coefficients of the log-cyclical components. You may want to recycle the Matlab code from Problem Set 4 to solve this question.

We will study Hansen's model of indivisible labor in combination with the concept of employment lotteries presented in Rogerson (1988)[2]. Assume that the objective function of an individual household is given by

$$\sum_{t=0}^{\infty} \beta^t \left[u(c_t) - v(h_t) \right],$$

where β is the objective discount factor, $u(c_t)$ is the momentary utility from consumption, and $v(h_t)$ denotes that household's disutility from working with $v'(\cdot) > 0$ and $v''(\cdot) > 0$. That is, working reduces utility and the more you work the more an additional hour hurts. In Hansen's model, households either work full time or not at all,

$$h_t \in \{0, 1\},$$

as labor is indivisible. The economy is populated by a continuum of ex-ante identical households of measure one. Here comes the model's tick: let's assume that all households agree to participate in a labor lottery. With probability ϕ_t they will have to work, and with probability $(1 - \phi_t)$ they don't. But no matter whether employed or unemployed, the households receive the same amount of consumption, c_t. Alternatively, you may think of a family whose members equally share consumption, but only some members work.

Since ex-ante (before the first labor lottery takes place) all households are identical, a planner would maximize the welfare function

$$\begin{aligned} \mathrm{E}\left\{ \sum_{t=0}^{\infty} \beta^t \left[u(c_t) - v(h_t) \right] \right\} &= \sum_{t=0}^{\infty} \beta^t \left[\phi_t \left[u(c_t) - v(1) \right] + (1 - \phi_t) \left[u(c_t) - v(0) \right] \right] \\ &= \sum_{t=0}^{\infty} \beta^t \left[u(c_t) - \phi_t \left[v(1) - v(0) \right] - v(0) \right]. \end{aligned}$$

Let us denote the difference in disutility of employment relative to unemployment by

$$d \equiv \left[v(1) - v(0) \right],$$

[2] Rogerson, R., "Indivisible Labor, Lotteries and Equilibrium," Journal of Monetary Economics, 21 (1988), 3-16.

and the fraction of the population that is working (the aggregate labor supply) as $l_t = \phi_t \times 1$. The welfare objective can be written as

$$\sum_{t=0}^{\infty} \beta^t \left[u(c_t) - l_t d\right],$$

where we have dropped the constant $-v(0)$ in the momentary utility which is not relevant to find the maximizer. Note that we started out with a utility function at the micro level that is non-linear in the individual labor supply, and ended up with a utility function that is linear in the aggregate labor supply. This implies that the labor supply elasticity at the macroeconomic level will be much higher than at the individual level.

Considering also the production side of the economy (where the stochastic level of productivity A_t drives the business cycle), the planner problem can be written in recursive form

$$V(k_t, A_t) = \max_{c_t, l_t, k_{t+1}} \left[u(c_t) - l_t d\right] + \beta \mathrm{E}_t V(k_{t+1}, A_{t+1}),$$

subject to

$$\begin{aligned} c_t + k_{t+1} &= A_t k_t^{\alpha} l_t^{1-\alpha} + (1-\delta)k_t \\ \log A_{t+1} &= \rho \log A_t + \sigma \varepsilon_{t+1}, \quad \rho \in (0,1), \quad \varepsilon_{t+1} \sim \mathcal{N}(0,1) \\ & c_t \geq 0, \quad l_t \in [0,1], \quad k_{t+1} \geq 0 \\ & k_0 > 0,\ A_0 > 0 \text{ given,} \end{aligned}$$

where E_t is short notation for the conditional expectation operator

$$\mathrm{E}_t x_{t+1} \equiv \mathrm{E}\left\{x_{t+1} | k_t, A_t\right\}.$$

Note that k_t denotes the aggregate capital stock, δ is the depreciation rate, and α is the capital income share in total production

$$y_t = A_t k_t^{\alpha} l_t^{1-\alpha}.$$

(b) At an interior solution, show that the optimality conditions of this planner problem can be written as

$$\begin{aligned} 0 &= u'(c_t) - \beta \mathrm{E}_t u'(c_{t+1}) \left[\alpha A_{t+1} k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} + (1-\delta)\right] \\ 0 &= u'(c_t)(1-\alpha) A_t k_t^{\alpha} l_t^{-\alpha} - d \\ 0 &= A_t k_t^{\alpha} l_t^{1-\alpha} + (1-\delta)k_t - c_t - k_{t+1} \\ 0 &= \rho \log A_t + \sigma \varepsilon_{t+1} - \log A_{t+1}. \end{aligned}$$

(c) Adding to the above derived optimality conditions rational expectations about the productivity process

$$\begin{aligned} \mathrm{E}_t \log A_{t+1} &= \rho \mathrm{E}_t \log A_t + \sigma \mathrm{E}_t \varepsilon_{t+1} \\ &= \rho \log A_t, \end{aligned}$$

the equilibrium can be written as

$$
\begin{aligned}
0 &= \mathrm{E}_t f\left(y_{t+1}, y_t, x_{t+1}, x_t\right) + \sigma[0,0,0,\varepsilon_{t+1}]' \\
&\equiv \mathrm{E}_t f\left(\begin{bmatrix} c_{t+1} \\ l_{t+1} \end{bmatrix}, \begin{bmatrix} c_t \\ l_t \end{bmatrix}, \begin{bmatrix} k_{t+1} \\ A_{t+1} \end{bmatrix}, \begin{bmatrix} k_t \\ A_t \end{bmatrix}\right) + \sigma[0,0,0,\varepsilon_{t+1}]' \\
&\equiv \mathrm{E}_t \begin{bmatrix} u'(c_t) - \beta u'(c_{t+1})\left[\alpha A_{t+1} k_{t+1}^{\alpha-1} l_{t+1}^{1-\alpha} + (1-\delta)\right] \\ u'(c_t)(1-\alpha) A_t k_t^{\alpha} l_t^{-\alpha} - d \\ A_t k_t^{\alpha} l_t^{1-\alpha} + (1-\delta)k_t - c_t - k_{t+1} \\ \log A_{t+1} - \rho \log A_t \end{bmatrix} + \sigma \begin{bmatrix} 0 \\ 0 \\ 0 \\ \varepsilon_{t+1} \end{bmatrix}.
\end{aligned}
$$

The solution to this rational expectations equilibrium will take the form

$$
\begin{aligned}
y_t &= g(x_t, \sigma) \\
x_{t+1} &= h(x_t, \sigma).
\end{aligned}
$$

In general, the equilibrium functions will be non-linear and have no closed form solutions. So, our strategy will be to approximate the equilibrium functions $g(x_t, \sigma)$ and $h(x_t, \sigma)$ around the non-stochastic steady state $(x_t = \bar{x}, \sigma = 0)$.

State the set of equilibrium equations that characterize the non-stochastic steady state

$$
0 = f(\bar{y}, \bar{y}, \bar{x}, \bar{x}), \quad \bar{y} = g(\bar{x}, 0).
$$

(d) Let marginal utility of consumption be of the form

$$
u'(c) = c^{-\gamma}, \quad \gamma > 0,
$$

and the disutility of labor of the form

$$
v(h) = \chi \frac{h^{1+1/\varphi}}{1+1/\varphi}, \quad \varphi > 0.
$$

Very often the solution to non-linear dynamic systems is approximated (to the first- or a higher-order) around the non-stochastic steady-state. The general recipe for the so-called log-linearization of a non-linear dynamic system of equations is:

(1) Take logs of the dynamic system, equation by equation.

(2) Do a first-order Taylor series expansion around a point (usually the non-stochastic steady state).

(3) Simplify terms so that all variables are expressed in percentage deviations from the linearization point.

Derive the log-linearization of this rational expectations model around the non-stochastic steady state and show that the linearized solution can be written in the form

$$\mathrm{E}_t \begin{bmatrix} \tilde{x}_{t+1} \\ \tilde{y}_{t+1} \end{bmatrix} = A \begin{bmatrix} \tilde{x}_t \\ \tilde{y}_t, \end{bmatrix}$$

where

$$\tilde{y}_t = \frac{y_t - \bar{y}}{\bar{y}}, \quad \tilde{x}_t = \frac{x_t - \bar{x}}{\bar{x}},$$

are variables expressed in deviations from the steady-state, and A is a 4-by-4 matrix.

(d) Now, let us calibrate the non-stochastic steady-state to some equilibrium values that seem reasonable (feel free to pick others if you want). We set the parameters that are standard in the literature (this is what researcher sometimes call external calibration) $\alpha = 1/3$, $\delta = 2.5/100$, $\gamma = 2$, and $\varphi = 2/3$. Let us target a real interest rate of 1% per quarter such that the capital-labor ratio is given by

$$1/100 = r - \delta = \alpha(\bar{k}/\bar{l})^{\alpha-1} - \delta \quad \Leftrightarrow \quad \bar{k}/\bar{l} = \left(\frac{\alpha}{1/100 + \delta}\right)^{1/(1-\alpha)}.$$

The steady-state Euler equation then implies that the discount factor consistent with this capital-labor ratio is given by

$$\beta = \left[\alpha(\bar{k}/\bar{l})^{\alpha-1} + (1-\delta)\right]^{-1}.$$

Let us also choose χ to target a labor supply $\bar{l} = 1/3$, meaning roughly that work time is 8 out of 24 hours per day. This implies that the steady-state capital stock will be given by

$$\bar{k} = \left(\frac{\alpha}{1/100 + \delta}\right)^{1/(1-\alpha)} \bar{l}.$$

Consumption will be

$$\bar{c} = \bar{k}^{\alpha}\bar{l}^{1-\alpha} - \delta\bar{k},$$

and the disutility of employment

$$d = (\bar{c})^{-\gamma}(1-\alpha)\bar{k}^{\alpha}\bar{l}^{-\alpha},$$

such that

$$v(1) - v(0) = \chi\frac{\varphi}{1+\varphi} - 0 = d \quad \Leftrightarrow \quad \chi = d\frac{1+\varphi}{\varphi}.$$

Given the calibrated parameters, compute the eigenvalues of the matrix A. According to Blanchard and Kahn (1980, Proposition 1), does the linearized system have a unique solution?