Lecture 2: Neoclassical Growth Theory
(*Acemoglu 2009, Chapter 8*)
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**Introduction**

- Ramsey or Cass-Koopmans model: differs from the Solow model insofar as it explicitly models the consumer side and endogenizes savings
- Beyond its use as a basic growth model, it is also a workhorse for many areas of macroeconomics
- Example: real and monetary business cycle theory
Preferences, Technology and Demographics I

- Infinite-horizon, continuous time.
- Representative household with instantaneous utility function
  \[ u(c(t)) , \]

Assumption \( u(c) \) is strictly increasing, concave, twice continuously differentiable with derivatives \( u' \) and \( u'' \), and satisfies the following Inada type assumptions:

\[ \lim_{c \to 0} u'(c) = \infty \quad \text{and} \quad \lim_{c \to \infty} u'(c) = 0. \]

- Suppose representative household represents set of identical households (normalized to 1)
- Each household has an instantaneous utility function \( u(c(t)) \)
- \( L(0) = 1 \) and
  \[ L(t) = \exp(nt) \]
Preferences, Technology and Demographics II

- All members of the household supply their labor inelastically
- Objective function of the representative household at $t = 0$:

$$U(0) \equiv \int_0^\infty \exp(-\rho t) \cdot L(t) \cdot u(c(t)) \, dt \quad (1)$$

$$= \int_0^\infty \exp(- (\rho - n) t) \cdot u(c(t)) \, dt,$$

where

- $c(t)$ = consumption per capita at $t$,
- $\rho$ = subjective discount rate, so that effective discount rate is $\rho - n$.

Objective function (1) embeds:

- Household is fully altruistic towards all of its future members, and makes allocations of consumption (among household members) cooperatively
- Strict concavity of $U(\cdot)$

Thus each household member has an equal consumption, $c(t) \equiv \frac{C(t)}{L(t)}$
Assumption: $\rho > n$

- Benchmark model without any technological progress
- Factor and product markets are competitive
- Production possibilities set of the economy is represented by

$$Y(t) = F[K(t), L(t)],$$

- $F$ features constant returns to scale and Inada conditions, i.e.,

$$Y = F_K \cdot K + F_L \cdot L \text{ (Euler Theorem)}$$

$$\lim_{K \to 0} F_K = \lim_{L \to 0} F_L = \infty, \quad \lim_{K \to \infty} F_K = \lim_{L \to \infty} F_L = 0$$
Preferences, Technology and Demographics IV

- Define variables in p.c. terms, $x(t) \equiv X(t)/L(t)$
- Per capita production function $f(\cdot)$

$$y(t) = F \left[ \frac{K(t)}{L(t)}, 1 \right] \equiv f(k(t)),$$

- Competitive factor markets imply:

$$R(t) = F_K[K(t), L(t)] = f'(k(t)).$$

and (from the Euler theorem)

$$w(t) = F_L[K(t), L(t)] = \frac{F[K(t), L(t)]}{L(t)} - F_K[K(t), L(t)] \frac{K(t)}{L(t)}$$

$$= f(k(t)) - k(t)f'(k(t)).$$
Denote asset holdings of the representative household at time $t$ by $A(t)$. Then,

$$\dot{A}(t) = r(t)A(t) + w(t)L(t) - c(t)L(t)$$

$r(t)$ is the market flow rate of return on assets, and $w(t)L(t)$ is the flow of labor income earnings of the household.

Defining per capita assets as

$$a(t) \equiv \frac{A(t)}{L(t)}$$

we obtain:

$$\dot{a}(t) = (r(t) - n)a(t) + w(t) - c(t)$$

Household assets can consist of capital stock, $K(t)$, which they rent to firms and bonds in zero net supply, $B(t)$. 
Given no uncertainty, arbitrage implies that the rate of return on bonds must equal the net return on capital (after depreciation at the rate $\delta$).

Both returns must equal $r(t) \Rightarrow$

$$r(t) = R(t) - \delta$$

Moreover, market clearing $\Rightarrow$

$$a(t) = k(t)$$
Let us return to the flow (or dynamic) budget constraint

$$\dot{a}(t) = (r(t) - n) a(t) + w(t) - c(t)$$

Imposing that the flow constraint holds for all $t \in [0, \infty[$ is not sufficient to ensure that a proper budget constraint hold unless we impose a lower bound on assets.

A dynasty could increase its consumption by running an ever growing debt.
The Budget Constraint II

- No-Ponzi Game Condition
- Total debt cannot grow at a rate exceeding the interest rate;

\[
\lim_{t \to \infty} A(t) \exp \left( - \int_0^t r(s) \, ds \right) \geq 0.
\]

- Equivalently, debt per capita cannot grow at a rate higher than \( r - n \):

\[
\lim_{t \to \infty} a(t) \exp \left( - \int_0^t (r(s) - n) \, ds \right) \geq 0.
\]

- Since it will never be optimal to have positive wealth asymptotically (formally, this will be captured by a Transversality Condition, TVC) the no-Ponzi-game condition can in fact be strengthened to:

\[
\lim_{t \to \infty} a(t) \exp \left( - \int_0^t (r(s) - n) \, ds \right) = 0.
\]
The no-Ponzi Game rules out the possibility for agents to borrow to finance present consumption and then use future borrowings to roll over the debt and pay the interest.

It can be shown formally (see textbook for a proof) that the no-Ponzi-game condition + period budget constraint ensures that the individual’s lifetime budget constraint holds in infinite horizon:

\[
\int_0^\infty c(t) \exp\left(-\int_0^t (r(s) - n) \, ds\right) \, dt = a(0) + \int_0^\infty w(t) \exp\left(-\int_0^t (r(s) - n) \, ds\right) \, dt
\]
Definition of Equilibrium

Definition  A competitive equilibrium of the Ramsey economy consists of paths \([C(t), K(t), w(t), R(t)]_{t=0}^\infty\), such that the representative household maximizes its utility given initial capital stock \(K(0)\) and the time path of prices \([w(t), R(t)]_{t=0}^\infty\), and all markets clear.

Notice:
the definition refers to the entire path of quantities and prices, not just steady-state equilibrium.
Household Maximization I

- Set up the current-value Hamiltonian:

\[ \hat{H}(a, c, \mu) = u(c(t)) + \mu(t) [w(t) + (r(t) - n) a(t) - c(t)], \]

- The solution must satisfy

1. **FOC**: \( \hat{H}_c(a, c, \mu) = 0 \)
   \[ \iff u'(c(t)) = \mu(t) \]

2. **EE**: \( \hat{H}_a(a, c, \mu) = -\dot{\mu}(t) + (\rho - n) \mu(t) = (r(t) - n) \mu(t) \)
   \[ \iff \frac{\dot{\mu}(t)}{\mu(t)} = -(r(t) - \rho) \]

3. **BC**: \( \dot{a}(t) = (r(t) - n) a(t) + w(t) - c(t) \)

4. **TVC**: \( \lim_{t \to \infty} [\exp(-(\rho - n) t) \cdot \mu(t) \cdot a(t)] = 0 \)
Household Maximization II

- Take logarithms in the FOC and differentiate with respect to time

\[
\frac{u''(c(t)) c(t) \dot{c}(t)}{u'(c(t)) c(t)} = \frac{\dot{\mu}(t)}{\mu(t)}.
\]

- Substituting into EE, obtain another form of the consumer Euler equation:

\[
\frac{\dot{c}(t)}{c(t)} = \frac{1}{\varepsilon_u(c(t))} (r(t) - \rho)
\]

where

\[
\varepsilon_u(c(t)) \equiv - \frac{u''(c(t)) c(t)}{u'(c(t))}
\]

is the elasticity of the marginal utility \( u'(c(t)) \).

- Consumption will grow over time when the discount rate is less than the rate of return on assets.
Household Maximization III

- Speed at which consumption will grow is related to the IES, elasticity of marginal utility of consumption, $\varepsilon_u(c(t))$.
- $\varepsilon_u(c(t))$ can also be interpreted (see book) as the inverse of the intertemporal elasticity of substitution (IES):
  - regulates willingness to substitute consumption over time.
- Suppose
  \[
  u(c) = \begin{cases} 
  \frac{c^{1-\theta} - 1}{1-\theta} & \text{if } \theta \neq 1 \text{ and } \theta \geq 0 \\
  \ln c & \text{if } \theta = 1
  \end{cases}
  \]
- This utility function (CRRA) induces a constant IES.
  - In particular, $\varepsilon_u(c(t)) = \theta$, so $1/\theta$ is the constant IES.
- CRRA is necessary to have balanced growth.
Under CRRA utility,
\[ \mu(t) = c(t)^{-\theta} \]
and the consumer Euler equation yields:
\[ \frac{\dot{\mu}(t)}{\mu(t)} = -(r(t) - \rho) = -\theta \frac{\dot{c}(t)}{c(t)} \quad \Rightarrow \quad \frac{\dot{c}(t)}{c(t)} = \frac{r(t) - \rho}{\theta} \]

Thus, integrating,
\[ \mu(t) = \mu(0) \exp \left( - \int_0^t (r(s) - \rho) \, ds \right), \]
Consider the TVC

\[ 0 = \lim_{t \to \infty} \left[ \exp \left( -(\rho - n) t \right) \cdot a(t) \cdot \mu(t) \right] \]

\[ = \lim_{t \to \infty} \left[ \exp \left( -(\rho - n) t \right) \cdot a(t) \cdot \mu(0) \exp \left( - \int_0^t (r(s) - \rho) \, ds \right) \right] \]

\[ = \lim_{t \to \infty} \left[ a(t) \exp \left( - \int_0^t (r(s) - n) \, ds \right) \right] \cdot c(0)^{-\theta}. \]

Thus, \( \lim_{t \to \infty} \left[ a(t) \exp \left( - \int_0^t (r(s) - n) \, ds \right) \right] = 0 \)

We can now provide an interpretation of the TVC
The transversality condition is a complementary condition that must hold (in standard problems) in order for the consumption/savings plan of the individual agents to be optimal.

In a finite-horizon problem, the TVC has a straightforward interpretation: the discounted value of the stock of capital left at the end of the planning period ($T$) must be zero

$$a(T) \cdot e^{-\int_0^T (r(v)-n) \, dv} = 0$$

As long as the interest rate is finite, the second term is positive, which reduces itself to the intuitive condition that $a_T = 0$. 
Transversality Condition II

- In the infinite horizon, we take the limit of the finite-horizon condition as $T$ tends to infinity:

$$\lim_{T \to \infty} \left[ a(T) \cdot e^{-\int_0^T (r(v)-n) \, dv} \right] = 0$$

- Interpretation: the PDV of assets at the “end of life” (infinity) must be zero. However, now $a(t)$ needs not converge to zero.

- A simple case in which the TVC holds is an economy converging to a steady-state where both $a(t)$ and $r(t)$ are constant.

- However, the TVC can also hold if $a(t) \to \infty$ as long as the second term goes to zero "sufficiently fast"
Equilibrium Prices I

- Equilibrium prices are given by

\[ R(t) = f'(k(t)) \quad \text{and} \quad w(t) = f(k(t)) - k(t)f'(k(t)). \]

- Since \( r(t) = R(t) - \delta \), then

\[ r(t) = f'(k(t)) - \delta. \]

- Substituting this into the consumer’s EE, we have

\[ \frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho}{\theta} \]

- Moreover, since \( a(t) = k(t) \) and \( \mu(t) = c(t)^{-\theta} \), the TVC can be written as

\[ \lim_{t \to \infty} \left[ \exp\left(- (\rho - n) t \right) \cdot \mu(t) \cdot a(t) \right] = 0 \]

\[ \lim_{t \to \infty} \left[ \exp\left(- (\rho - n) t \right) \cdot c(t)^{-\theta} \cdot k(t) \right] = 0 \]
Finally, let us go back to the individual budget constraint

\[ \dot{a}(t) = (r(t) - n) a(t) + w(t) - c(t) \]

And using the equilibrium conditions

\[
\begin{align*}
    a(t) &= k(t) \\
    r(t) &= f'(k(t)) - \delta \\
    w(t) &= f(k(t)) - k(t)f'(k(t))
\end{align*}
\]

We conclude that

\[ \dot{k}(t) = f(k(t)) - (n + \delta)k(t) - c(t), \]

that can be interpreted as an aggregate resource constraint.
In an economy that admits a representative household, optimal growth involves maximization of utility of representative household subject to technology and feasibility constraints:

$$\max_{[k(t), c(t)]_{t=0}^\infty} \int_0^\infty \exp(-(\rho - n) t) u(c(t)) \, dt,$$

subject to

$$\dot{k}(t) = f(k(t)) - (n + \delta)k(t) - c(t),$$

and $k(0) > 0$.

Versions of the First and Second Welfare Theorems for economies with a continuum of commodities: solution to this problem should be the same as the equilibrium growth problem.

Let us show the equivalence directly.
Again set up the current-value Hamiltonian:

\[ \tilde{H}(k, c, \mu) = u(c(t)) + \mu(t) \left[ f(k(t)) - (n + \delta)k(t) - c(t) \right], \]

The solution must satisfy

\[ FOC_{PL} : \quad \tilde{H}_c(k, c, \mu) = 0 \]
\[ \iff \quad u'(c(t)) = \mu(t) \]

and \( EE_{PL} \):

\[ \tilde{H}_k(k, c, \mu) = -\dot{\mu}(t) + (\rho - n)\mu(t) = (f'(k(t)) - \delta - n)\mu(t) \]
\[ \iff \quad \frac{\dot{\mu}(t)}{\mu(t)} = -(f'(k(t)) - \delta - \rho) \]

\( RC \): \quad \dot{k}(t) = f(k(t)) - (n + \delta)k(t) - c(t) \]

\( TVC_{PL} \): \quad \lim_{t \to \infty} \left[ \exp \left( - (\rho - n) t \right) \cdot \mu(t) \cdot k(t) \right] = 0
Assume CRRA. Repeating the same steps as before,

\[
\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho}{\theta},
\]

\[
\dot{k}(t) = f(k(t)) - (n + \delta)k(t) - c(t),
\]

\[
\lim_{t \to \infty} \left[ \exp \left( - (\rho - n) t \right) \cdot c(t)^{-\theta} \cdot k(t) \right] = 0
\]

which are identical to the laissez-faire equilibrium conditions.

Thus the competitive equilibrium is a Pareto optimum and the Pareto allocation can be decentralized as a competitive equilibrium.
Steady-State Equilibrium I

- Steady-state equilibrium is defined as an equilibrium path in which capital-labor ratio, consumption and output are constant. Thus:

\[
\frac{\dot{c}(t)}{c(t)} = \frac{f'(k^*) - \delta - \rho}{\theta} = 0
\]

\[\iff f'(k^*) = \rho + \delta\]

- Pins down the steady-state capital-labor ratio only as a function of the production function, the discount rate and the depreciation rate.

- Then, the resource constraint pins down the steady-state consumption level:

\[
\dot{k}(t) = f(k(t)) - (n + \delta)k(t) - c(t) = 0
\]

\[\iff c^* = f(k^*) - (n + \delta)k^*.
\]
A steady state where the capital-labor ratio and thus output are constant necessarily satisfies the TVC:

$$\lim_{{t \to \infty}} \left[ \exp \left( - (\rho - n) t \right) \cdot k^* \cdot (c^*)^{-\theta} \right] = 0$$

which is true as long as $\rho > n$. 
Equilibrium is determined by two differential equations:

\[
\dot{k}(t) = f(k(t)) - (n + \delta)k(t) - c(t)
\]

\[
\frac{\dot{c}(t)}{c(t)} = \frac{f'(k(t)) - \delta - \rho}{\theta}
\]

plus an initial condition, \(k(0) > 0\), and a terminal condition:

\[
\lim_{t \to \infty} \left[ \exp\left( - (\rho - n) t \right) \cdot k(t) \cdot (c(t))^{-\theta} \right] = 0.
\]
Appropriate notion of *saddle-path stability*:

- $c$ (or, equivalently, $\mu$) is the control variable, and $c(0)$ (or $\mu(0)$) is free: it has to adjust to satisfy transversality condition.
- If there were more than one path equilibrium would be indeterminate.

Economic forces are such that indeed there will be a one-dimensional manifold of stable solutions tending to the unique steady state.

See Figure.
Figure: Transitional dynamics in the baseline neoclassical growth model
Intuitive argument:

- If $c(0)$ started below it, say $c''(0)$, consumption would reach zero, thus capital would accumulate continuously until the maximum level of capital (reached with zero consumption). This would violate the transversality condition.
- If $c(0)$ started above this stable arm, say at $c'(0)$, the capital stock would reach 0 in finite time, while consumption would remain positive. But this would violate feasibility (a little care is necessary with this argument, since necessary conditions do not apply at the boundary).
Technological Change and the Canonical Neoclassical Model I

- Extend the production function to:

\[ Y(t) = F[K(t), A(t)L(t)], \]

where

\[ A(t) = \exp(gt) A(0). \]

- Note: we assume labor-augmenting technological change. Else, there would be no balanced growth equilibrium.
Define $\hat{x}(t) \equiv X(t) / (A(t) L(t))$

$$\hat{y}(t) = F \left[ \frac{K(t)}{A(t) L(t)}, 1 \right] \equiv f(\hat{k}(t)),$$

- Assume CRRA preferences
The equilibrium is now fully characterized by the following dynamic equations

\[
\frac{\dot{c}(t)}{\hat{c}(t)} = \frac{1}{\theta} \left( f'(\hat{k}(t)) - \delta - \rho - \theta g \right),
\]

\[
\dot{k}(t) = f(\hat{k}(t)) - (n + g + \delta) \hat{k}(t) - \hat{c}(t),
\]

plus an initial condition, \( \hat{k}(0) > 0 \), and a terminal condition (TVC)

\[
= \lim_{t \to \infty} \left\{ \exp \left( - (\rho - n - (1 - \theta) g) t \right) \cdot \hat{k}(t) \cdot (\hat{c}(t))^{-\theta} \right\} = 0.
\]
Equilibrium (derivation EE, see book for more)

\[ r(t) = f'(\hat{k}(t)) - \delta \]

- Since \( c(t) = A(t) \cdot \hat{c}(t) \), then
  \[ \frac{\dot{c}(t)}{c(t)} = \frac{\dot{c}(t)}{\hat{c}(t)} + g \]

- Then:
  \[ \frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta} (r(t) - \rho) \iff \frac{\dot{c}(t)}{\hat{c}(t)} = \frac{1}{\theta} (f'(\hat{k}(t)) - \delta - \rho - \theta g) \]
In steady state, $f'(\hat{k}^*) = \rho + \delta + \theta g$.

Pins down the steady-state value of the normalized capital ratio $\hat{k}^*$ uniquely.

Normalized consumption level is then given by

$$\hat{c}^* = f(\hat{k}^*) - (n + g + \delta) \hat{k}^*,$$

Per capita consumption grows at the rate $g$.

The TVC now requires $\rho - n > (1 - \theta) g$. 
Proposition Consider the neoclassical growth model with labor augmenting technological progress at the rate $g$ and CRRA preferences. Suppose that $\rho - n > (1 - \theta) g$. Then there exists a unique balanced growth path with a normalized capital to effective labor ratio of $\hat{k}^*$, given by $f' (\hat{k}^*) = \rho + \delta + \theta g$, and output per capita and consumption per capita grow at the rate $g$. 
Transitional dynamics

Figure: When $g > 0$, simply replace $k$ and $c$ by $\hat{k}$ and $\hat{c}$
Comparative Dynamics I

- Comparative statics: changes in steady state in response to changes in parameters.
- Comparative dynamics look at how the entire equilibrium path of variables changes in response to a change in policy or parameters.
- E.g.: Initial steady state represented by \((k^*, c^*)\). Unexpectedly, discount rate declines to \(\rho' < \rho\).
- Following the decline \(\hat{c}^*\) is above the stable arm of the new dynamic system: consumption must drop immediately.
**Figure:** The dynamic response of capital and consumption to a decline in the discount rate from $\rho$ to $\rho' < \rho$. 
Introduce linear tax policy: returns on capital net of depreciation are taxed at the rate $\tau$ and the proceeds of this are redistributed back to the consumers.

Capital accumulation equation remains as above:

$$\dot{k}(t) = f(\hat{k}(t)) - \hat{c}(t) - (n + g + \delta) \hat{k}(t),$$

But interest rate faced by households changes to:

$$r(t) = (1 - \tau) (f'(\hat{k}(t)) - \delta),$$
Growth rate of normalized consumption is then obtained from the consumer Euler equation

\[
\frac{\dot{c}(t)}{c(t)} = \frac{1}{\theta} (r(t) - \rho - \theta g).
\]

This implies

\[
f'(\hat{k}^*) = \delta + \frac{\rho + \theta g}{1 - \tau}.
\]

Since \(f'(\cdot)\) is decreasing, higher \(\tau\), reduces \(\hat{k}^*\).

Higher taxes on capital have the effect of depressing capital accumulation and reducing income per capita.
Appraisal neoclassical model

- Major contribution: open the black box of capital accumulation by specifying the preferences of consumers.
- Also by specifying individual preferences we can explicitly compare equilibrium and optimal growth.
- Paves the way for further analysis of capital accumulation, human capital and endogenous technological progress.
- However, this model, by itself, does not enable us to answer questions about the fundamental causes of economic growth.
- But it clarifies the nature of the economic decisions so that we are in a better position to ask such questions.
AK model I

- Neoclassical model: no autonomous engine of growth. In the absence of exogenous trend, growth dies off in the long-run.
  1. No theory of determinants of long-run growth;
  2. No theory of determinants of long-run cross-country differences in growth rates;

- The AK-Model is a very simple model that can be viewed as the "limit case" of the neoclassical growth model. It provides the common analytical framework for a number of more interesting applications.
AK model II

- Production technology (g=0):

\[ f(k) = Ak \]

- Equilibrium interest rate is \( r(t) = A - \delta \).

- Assume CRRA utility

- Given \( k(0) \), a competitive equilibrium is determined by

\[
\begin{align*}
\dot{c}(t) &= \frac{A - \delta - \rho}{\theta} \cdot c(t) \\
\dot{k}(t) &= Ak(t) - c(t) - (\delta + n)k(t)
\end{align*}
\]

\( (EE) \quad (BC) \)

\[
\lim_{t \to \infty} \left[ \exp \left( - (\rho - n) t \right) \cdot k(t) \cdot \left( c(t) \right)^{-\theta} \right] = 0
\]

\( (TVC) \)
AK model III

We can obtain an explicit analytical solution:

(a) Guess a steady-state solution such that \( c/k \) is constant (assume \( A > \delta + \rho \)).

\[
\frac{\dot{c}(t)}{c(t)} = \frac{\dot{k}(t)}{k(t)} = \frac{A - \delta - \rho}{\theta} = \gamma
\]

(b) Use (BC)

\[
\frac{\dot{k}(t)}{k(t)} = (A - \delta - n) - \frac{c(t)}{k(t)}.
\]

(c) From (a)+(b):

\[
\frac{c(t)}{k(t)} = \frac{c}{k} = \rho - n - \frac{1 - \theta}{\theta} \cdot [A - \delta - \rho].
\]

In particular:

\[
c(0) = \left\{ \rho - n - \frac{1 - \theta}{\theta} \cdot [A - \delta - \rho] \right\} k(0).
\]
AK model IV

- Hence (a) and the solution for \( c(0) \), we obtain analytical solutions:

\[
c(t) = c(0) \cdot \exp \left( \left( \frac{A - \delta - \rho}{\theta} \right) t \right)
\]

\[
k(t) = k(0) \cdot \exp \left( \left( \frac{A - \delta - \rho}{\theta} \right) t \right)
\]

- TVC (after replacing \( k(t) \) by its solution):

\[
0 = \lim_{t \to \infty} \left[ \exp \left( - (\rho - n) t \right) \cdot \frac{k(t)}{c(t)} \cdot c(t) \cdot (c(t))^{-\theta} \right]
\]

\[
= \lim_{t \to \infty} \left[ \exp \left( - (\rho - n) t \right) \cdot \left( \frac{c}{k} \right)^{-1} \cdot (c(0))^{1-\theta} \right.
\]

\[
\left. \cdot \exp \left( \left( \frac{1-\theta}{\theta} (A - \delta - \rho) \right) t \right) \right]
\]

provided that the following condition (bounded utility) holds:

\[
\rho > n + (1 - \theta) (A - \delta - n)
\]

- No transitional dynamics.
In this model, policies have permanent effects on growth.

Consider again the introduction of a permanent tax on the returns on capital. The proceeds are rebated as lump-sums.

The equilibrium interest rate is \( r = (1 - \tau) (A - \delta) \), and the equilibrium growth rate is:

\[
\gamma_\tau = \frac{(1 - \tau) (A - \delta) - \rho}{\theta}
\]
Two Simple AK Models

- Two simple models that deliver AK dynamics
- Assume \( n = g = 0 \)
- Basic human capital and knowledge spillovers
Suppose agents can accumulate both physical and human capital. Let the technology be

\[ Y = F(K, H) = AK^{\alpha} H^{1-\alpha} = AK \left( \frac{H}{K} \right)^{1-\alpha} \]
Basic Human Capital Model II

• Assume (unrealistically):

1. Physical capital, human capital and consumption goods are produced with the same technology: One unit of final output can be used for consumption, investment in physical capital and investment in human capital.
2. All investments are fully reversible.
3. Same depreciation (rate $\delta$) for both types of capital (unimportant).
Basic Human Capital Model III

- No arbitrage implies: \( R_K = R_H = r + \delta. \)
- Firms’ profit-maximization:

\[
R_K = \alpha A \left( \frac{H}{K} \right)^{1-\alpha} = (1 - \alpha) A \left( \frac{H}{K} \right)^{-\alpha} = R_H
\]

Solving for human-to-physical ratio yields:

\[
\frac{H}{K} = \frac{1 - \alpha}{\alpha}
\]

- Hence, substituting away \( H/K \):

\[
\begin{align*}
  r &= \alpha^\alpha (1 - \alpha)^{1-\alpha} A - \delta, \\
  Y &= \left( \frac{1 - \alpha}{\alpha} \right)^{1-\alpha} AK
\end{align*}
\]
The equilibrium features

\[
\dot{c}(t) = \frac{\alpha^\alpha (1 - \alpha)^{1-\alpha} A - \delta - \rho}{\theta} \cdot c(t),
\]

\[
\dot{k}(t) = \left(\frac{1 - \alpha}{\alpha}\right)^{1-\alpha} Ak(t) - c(t) - \delta k(t),
\]

plus a TVC

In equilibrium, the economy grows at the constant rate

\[
\gamma = \frac{\alpha^\alpha (1 - \alpha)^{1-\alpha} A - \delta - \rho}{\theta}.
\]
Learning-by-doing Externalities I

- External effects of capital accumulation on productivity.
- As capital in firm $i$ accumulates, it has a productivity-enhancing effect on the capital installed in all firms.
- It becomes important to distinguish between firm-level and aggregate variables.
- Firm-level technology:

$$Y_i = F(K_i, \tilde{A}L_i)$$

- Labor-augmenting technical progress ($\tilde{A}$) is not firm-specific. We assume

$$\tilde{A} = \phi K$$
Learning-by-doing Externalities II

- $\tilde{A}$ can be interpreted as public knowledge. Knowledge is assumed to have a non-rival character: when a firm adds to the stock of knowledge, all firms in the economy can benefit from this addition.
- Knowledge accumulation is assumed to be a pure spillover.
- For simplicity, we restrict attention to Cobb-Douglas technology:

$$F(K_i, \tilde{A}L_i) = K_i^\alpha (\tilde{A}L_i)^{1-\alpha} = AK_i^\alpha (KL_i)^{1-\alpha}$$

where $A \equiv \phi^{1-\alpha}$
Firms takes $\bar{K}$ as parametric. Thus, the equilibrium rates of return are:

$$R = F_{K_i} (K_i, \tilde{A} L_i) = \alpha A \left( \frac{K L_i}{K_i} \right)^{1-\alpha} \quad \text{(LBD-FOC1)}$$

$$w = F_{L_i} (K_i, \tilde{A} L_i) = \frac{(1 - \alpha) A K_i^\alpha (K L_i)^{1-\alpha}}{L_i} \quad \text{(LBD-FOC2)}$$

Assume a continuum of identical firms with total measure equal to $M$. Thus, in a symmetric equilibrium,

$$MK_i = K \text{ and } ML_i = L. \quad \text{(LBD-EQ)}$$
IMPORTANT: to characterize the competitive equilibrium, one must substitute (LBD-EQ) into (LBD-FOC1)-(LBD-FOC2) (i.e., atomistic firms ignore the effect of their investments on aggregate productivity).

I.e., firms act in an uncoordinated fashion. So, using (LBD-EQ) to eliminate \( K_i \) and \( L_i \), leads to:

\[
\begin{align*}
  r &= R - \delta = \alpha A L^{1-\alpha} - \delta \\
  w &= \frac{(1 - \alpha) A K}{L^\alpha} = (1 - \alpha) \cdot A \cdot k \cdot L^{1-\alpha}
\end{align*}
\]
Learning-by-doing Externalities V

- The equilibrium conditions are

\[ \begin{align*}
\dot{c}(t) & = \frac{\alpha A L^{1-\alpha} - \delta - \rho}{\theta} \cdot c(t) \\
\dot{k}(t) & = (A L^{1-\alpha} - \delta) k(t) - c(t)
\end{align*} \]

plus a TVC

- The dynamics of this model are isomorphic to those of the AK model. But there are two differences:
  1. Scale effects
  2. Equilibrium is not Pareto optimal
     (discussed as an exercise).