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Exam
Advanced Macroeconomic Theory ECON 5300

Good Luck!

1 OLG Model

Consider the two-period OLG economy discussed in the lecture with positive population growth, i.e., $n > 0$ and positive technological growth rate $g > 0$. Denote the depreciation rate as δ. The household's instantaneous utility function is CRRA: $U(C) = C^(1-\sigma)/(1-\sigma)$ and time discount rate is ρ, i.e., the lifetime utility is

$$\frac{C_y^{1-\sigma}}{1-\sigma} + \frac{1}{1+\rho}\frac{C_o^{1-\sigma}}{1-\sigma}.$$

The production function of the firm is Cobb-Douglas: $F(K,L) = K^{\alpha}(AL)^{1-\alpha}$.

1. [10 marks] Write down the household's problem and solve for household savings, as a function of wage and interest rate. How do savings respond to an increase of interest rate? Explain the answer.
   ANS:

$$\begin{aligned} \max & \frac{C_y^{1-\sigma}-1}{1-\sigma} + \frac{1}{1+\rho}\frac{C_o^{1-\sigma}-1}{1-\sigma} \\ \text{s.t. } & C_y + S = W, \\ & C_o = (1+r)\,S. \end{aligned}$$

$$S = \frac{W}{1+(1+\rho)^{1/\sigma}\,(1+r')^{1-1/\sigma}}.$$

   S increases when r' increases if $1-1/\sigma < 0$, i.e., $\sigma < 1$.

2. Derive the dynamic equations that characterize aggregate capital K and capital per efficient unit of labour $k = K/(AL)$, where A is the technology level and L is population. An implicit equation is sufficient.
   ANS:

$$\begin{aligned} k' &= \frac{S\left(f(k) - kf_k(k), f_k(k') - \delta\right)}{(1+n)(1+g)} \\ &= \frac{(1-\alpha)\,k^{\alpha}}{\left(1+(1+\rho)^{1/\sigma}\left(1+\alpha\,(k')^{\alpha-1}-\delta\right)^{1-1/\sigma}\right)(1+n)(1+g)}. \end{aligned}$$

3. Suppose that the economy is on its balanced growth path before period t_0. Unexpectedly, at the beginning of period t_0, the population jumps to a higher level $L' > L(t_0)$ while population growth rate afterwards decreases to $n' < n$ and stays at the low level forever. How do aggregate output Y and output per efficient unit of labour $y = Y/(AL)$ from t_0 on respond, respectively? How does the long-run y compare to the initial y at period t_0? Explain your answers.
   ANS: Y at time t_0 jumps up while y drops. (higher L increases Y but capital is predetermined so K/L and y fall). In the long run, y is higher than the initial y at period t_0 since over time K accumulates until the old steady-state value of K/L is reached.

4. Suppose $\sigma = 1, n = g = 0$, and the economy is on its balanced growth path before period t_0. Unexpectedly, at the beginning of period t_0, the production function changes, α drops to a lower level $\alpha' < \alpha$ and stays at the new level forever. How does the long-run level of k change? How does k converges to the long-run level? Explain your answers.
ANS: The solution is as follows:

$$k' = \frac{(1-\alpha)\, k^{\alpha}}{2+\rho}.$$

Steady state value k^*:

$$k^* = \frac{(1-\alpha)\, (k^*)^{\alpha}}{2+\rho},$$
$$k^* = \left(\frac{1-\alpha}{2+\rho}\right)^{\frac{1}{1-\alpha}},$$

We can compute whether $(1-\alpha)\, k^{\alpha}$ increase or decrease with α by:

$$\begin{aligned}
\frac{d \log\left((1-\alpha)\, k^{\alpha}\right)}{d\alpha} &= \frac{d \log\left((1-\alpha)\right)}{d\alpha} + \frac{d\alpha \log\left(k\right)}{d\alpha} \\
&= -\frac{1}{1-\alpha} + \log k \\
&= -\frac{1}{1-\alpha} + \frac{1}{1-\alpha} \log\left(\frac{1-\alpha}{2+\rho}\right) \\
&< 0.
\end{aligned}$$

So k' decreases.
This implies that k^* also decreases. We can formally show by computing

$$\begin{aligned}
\frac{d \log k^*}{d\alpha} &= \frac{d \frac{1}{1-\alpha} \log\left(\frac{1-\alpha}{2+\rho}\right)}{d\alpha} \\
&= \frac{1}{(1-\alpha)^2} \log\left(\frac{1-\alpha}{2+\rho}\right) - \frac{1}{1-\alpha} \frac{2+\rho}{1-\alpha} \\
&< 0.
\end{aligned}$$

2 Value Function

Consider the following decentralised economy where the representative agent makes per capita consumption c_t and per capita labour supply l_t decisions,

$$max_{c_t,l_t} E_0 \sum_{t=0}^{\infty} \beta^t u(c_t, 1 - l_t)$$

subject to

$$a_{t+1} = r_t a_t + w_t l_t - c_t,$$

where $u(.,.)$ is a per period utility function that satisfied the Inada conditions, $\beta \in [0, 1]$ denotes the discount factor, a_t the level of assets, r_t the real return to asset holdings and w_t the hourly wage. Assume that all markets operate under perfect competition and aggregate production is given by $Y_t = z_t K_t^{\alpha} N_t^{1-\alpha}$, where K_t denotes the stock of capital and N_t the total labour supply. Note that this formulation implies that per capita production is given by $y_t = z_t k_t^{\alpha}$, where $\alpha \in [0, 1]$ and z_t is a random technology shock that is known at the beginning of period t and E_0 is the expectation at time $t = 0$ based on the distribution of z. Business cycles are generated by z_t.

1. Using Bellman's principle of optimality write the above optimization problem in terms of its value function, $V(a_t, z_t)$.

   ANS:

   $$V(a_t, z_t) = max_{c_t,l_t}[u(c_t, 1 - l_t) + E_t V(a_{t+1}, z_{t+1})]$$

   subject to

   $$a_{t+1} = r_t a_t + w_t l_t - c_t,$$

2. From the derived Bellman equation and using the envelope theorem, show that the first-order condition that describe the optimal choices of consumption and leisure are given by

   $$u_1(c_t, 1 - l_t) = \beta E_t[u_1(c_{t+1}, 1 - l_{t+1}) r_{t+1} | z_t]$$
   $$u_2(c_t, 1 - l_t) = u_1(c_t, 1 - l_t) w_t,$$

   where $u_1(.)$ refers to the first derivative of u with respect to c and $u_2(.)$ refers to the first derivative of u with respect to l. Explain the economic intuition behind these equations.

   ANS:

   $$c_t: \quad u_1(c_t, 1 - l_t) = \beta E_t[V_a(a_{t+1}, z_{t+1})]$$
   $$a_t: \quad V_a(a_t, z_t) = r_t u_1(c_t, 1 - l_t)$$
   $$l_t: \quad u_2(c_t, 1 - l_t) = u_1(c_t, 1 - l_t) w_t,$$

3. Consider the case in which $ln(c_t) + b\frac{(1-l_t)^{(1-\gamma)}}{(1-\gamma)}$, where $\gamma > 0$ the elasticity of intertemporal substitution and $b > 0$. Using the Euler equation for consumption derived earlier, show that the savings rate is constant.

   ANS:

Guess $c_t = \mu y_t$ and use the Euler equation

$$\frac{\mu y_{t+1}}{\mu y_t} = \beta\alpha y_{t+1}/k_{t+1}, \quad y_{t+1}/k_{t+1} = (k_{t+1})^{\alpha-1}.$$

which implies that $k_{t+1} = \alpha\beta y_t$. Using this in the resource constraint yields

$$N_{t+1}k_{t+1} = K_{t+1} = Y_t - C_t = N_t(1-\mu)y_t$$

such that

$$K_{t+1} = N_{t+1}k_{t+1} = N_{t+1}\alpha\beta y_t = N_t(1-\mu)y_t$$

such that

$$1 - \mu = 1 - \alpha\beta(1+g_N)$$

and

$$\mu = \alpha\beta(1+g_N)$$

Note: In marking the essays it should be noted that, due to a mistake, the problem set presented to the students during the exam contained a certain fraction of the solution sketch. This could have caused some confusion and may have affected the response. Also, the problem set was not a complete replica of problems presented in class. The grading will take this into account.