Lecture 2 - Auctions

Auctions are widely used in procurement and in sales of:

- assets from public to private owners (privatizations),
- rights to use natural resources,
- bonds from public utilities,
- commodities,
- consumer goods (e.g. online auctions),
- objects of art.

Definition. Auction are universal and anonymous mechanisms.

Auctions are used whenever the seller does not know the potential buyers' valuations. We focus on auctions for a single indivisible object.

If each bidder knows the value of the object to herself at the time of bidding, the situation is called an *auction with private values*. Implicit here: value to other bidders is not perfectly known (hard to justify why a bidder knows the other bidders' valuations, but the seller does not); anyway knowledge of the other bidders' valuations would not affect how much a bidder values the object in this setting.

If values are unknown at the time of bidding and might be affected by information available to other bidders, then we have an *auction with interdependent values* (a special case of this is when the value is the same to all bidders, thus we have a *common value auction*).

Note that interdependence here refers only to valuations, not to information: we can have cases in which values are interdependent but the signals received by the bidders are statistically independent, and we can have situations in which signals are correlated, but valuations are independent.

Common forms: *English*, or *open ascending auction*; *Dutch* or *open descending auction*; *sealed-bid first-price auction*; *sealed-bid second-price auction*.

Note that:

(1) first price auction and Dutch auction are essentially equivalent in terms of strategies and equilibrium outcomes.

(2) second price auction and English auction are essentially equivalent *when values are private*.

Standard questions in auction theory:

- Efficiency: does the auction allocate the object to the bidder with the highest valuation?
- Optimality: does the auction maximize the revenue of the seller?

1. Auctions with Independent Values

Setting #1:

- N bidders, submitting their bids simultaneously.
- The object sold has value X_i to bidder i.
- Each X_i is independently and identically distributed on some interval $[0, \omega]$ according to some increasing CDF $F(\cdot)$; F has a continuous density $f \equiv F'$ and has full support.
- Each bidder i knows the realization x_i of random variable X_i and knows how the values of the other bidders are distributed.
- Bidders have no liquidity or budget constraints.
- All bidders are risk-neutral.

Definition. *An auction is standard if the rules of the auction dictate that the person who bids the highest amount is awarded the object.*

Lotteries are auctions, but they are not standard auctions. Auctions with a reserve price (a minimum bid below which there the object is not sold) are not standard auctions.

Definition. *An equilibrium of an auction A is symmetric if all bidders bid according to the same function $\beta^A : [0, \omega] \to \mathbb{R}$. An equilibrium of an auction is increasing if $\beta^A(\cdot)$ is increasing.*

Theorem. *(**Revenue Equivalence Principle**) In setting #1, any symmetric and increasing equilibrium of any standard auction such that the expected payment of a bidder with value 0 is 0, yields the same expected revenue to the seller.*

Proof. Fix a symmetric equilibrium. Let $m^A(x)$ denote the equilibrium expected payment by a bidder with valuation x. Suppose every bidder $i \neq 1$ bids according to β^A. Suppose that function β^A and auction rules are such that $m^A(0) = 0$.

Consider the expected payoff of bidder 1 when her valuation is x and she bids $\beta^A(z)$ for some $z \in [0, \omega]$.

As the auction is standard and the equilibrium is increasing, bidder 1 wins if $\beta^A(z) > \beta^A(Y_1)$ where $Y_1 = \max\{x_2, .., x_N\}$. Let $G(\cdot)$ denote the CDF of Y_1 and $g(\cdot) = G'(\cdot)$. Then bidder 1's expected payoff is equal to:

$$\Pi(z, x) = G(z)x - m^A(z).$$

In equilibrium, the first order condition for bidder 1 must hold for $z = x$:

$$\frac{\partial \Pi(z, x)}{\partial z}|_{z=x} = \left(g(z)x - \frac{dm^A(z)}{dz} \right)|_{z=x} = g(x)x - \frac{dm^A(x)}{dx} = 0.$$

In equilibrium, this condition must hold for any $x \in [0, \omega]$, thus, for any $x \in [0, \omega]$:

$$m^A(x) = m^A(0) + \int_0^x \frac{dm^A(y)}{dy} dy = m^A(0) + \int_0^x yg(y)dy = \int_0^x yg(y)dy.$$

As $m^A(\cdot)$ does not depend on the specific form of the auction, this completes the proof. QED

Increasing equilibria in standard auction ensure efficiency. Nevertheless, it can be optimal for a seller to use a non-standard auction format, such as an auction with reserve prices.

Example.

Remark. Symmetric equilibrium strategies in a first price auction require each bidder i to bid

$$\beta^I(x_i) = E[Y_i | Y_i < x_i] = \frac{1}{G(x_i)} \int_0^{x_i} g(y)ydy = x_i - \int_0^{x_i} \frac{G(y)}{G(x_i)} dy.$$

To check this remark holds, suppose all other bidders bid according to this strategy. Then a bid $b > \beta^I(\omega)$ cannot be optimal. For any bid $0 \leq b \leq \beta^I(\omega)$, denote z the valuation for which

$\beta^I(z) = b$. Bidder i's expected payoff from bidding $\beta^I(z)$ when her value is x_i corresponds to:

$$\begin{aligned}
\Pi(\beta(z), x_i) &= G(z)\left[x_i - \beta(z)\right] \\
&= G(z)x_i - G(z)E\left[Y_i | Y_i < z\right] \\
&= G(z)x_i - G(z)\frac{1}{G(z)}\int_0^z g(y)ydy \\
&= G(z)x_i - \int_0^z g(y)ydy
\end{aligned}$$

Using integration by parts:

$$\Pi(\beta(z), x_i) = G(z)x_i - zG(z) + \int_0^z G(y)dy = G(z)\left(x_i - z\right) + \int_0^z G(y)dy.$$

Therefore

$$\begin{aligned}
\Pi(\beta(x_i), x_i) - \Pi(\beta(z), x_i) &= G(z)\left(z - x_i\right) - \int_{x_i}^z G(y)dy \\
&= \int_{x_i}^z \left(G(z) - G(y)\right)dy.
\end{aligned}$$

Note that the last expression is non-negative regardless of whether z is smaller or larger than x_i, thus proving the remark.

Suppose that the auctioneer imposes a reserve price $r \in (0, \omega)$. Then there is an equilibrium in which each bidder i bids the largest value between r and $E\left[Y_i | Y_i < x_i\right]$, thus

$$\begin{aligned}
\beta^I(x_i) &= E\left[\max\{r, Y_i\} | Y_i < x_i\right]. \\
&= \frac{1}{G(x_i)}\left(\int_r^{x_i} g(y)ydy + G(r)r\right)
\end{aligned}$$

The expected payment of a bidder i with $x_i \geq r$ is

$$\begin{aligned}
m^I(x_i, r) &= G(x_i)\beta^I(x_i) \\
&= rG(r) + \int_r^{x_i} yg(y)dy
\end{aligned}$$

Thus, the ex-ante expected payment of a bidder is:

$$\begin{aligned}
E\left[m^I(X,r)\right] &= \int_r^\omega m^I(x,r)f(x)dx \\
&= rG(r)\left(1-F(r)\right)+\int_r^\omega f(y)\int_r^y xg(x)dxdy \\
&= rG(r)\left(1-F(r)\right)+F(y)\int_r^y xg(x)dx|_r^\omega-\int_r^\omega F(y)yg(y)dy \\
&= rG(r)\left(1-F(r)\right)+\int_r^\omega xg(x)dx-\int_r^\omega F(y)yg(y)dy \\
&= rG(r)\left(1-F(r)\right)+\int_r^\omega yg(y)\left(1-F(y)\right)dy
\end{aligned}$$

Suppose the seller attaches value 0 to the object (the case of sellers' value $x_0 \in (0,\omega)$ is a straightforward extension, see K page 22), then the overall expected payoff of the seller is:

$$\Pi_0 = N \times E\left[m^A\left(X,r\right)\right]$$

Differentiate with respect to r:

$$\begin{aligned}
\frac{d\Pi_0}{dr} &= N\times\left[G(r)\left(1-F(r)\right)+rg(r)\left(1-F(r)\right)-rG(r)f(r)-rg(r)\left(1-F(r)\right)\right] \\
&= N\times\left[1-F(r)-rf(r)\right]G(r)
\end{aligned}$$

Using the hazard rate $\lambda(x)=\frac{f(x)}{1-F(x)}$:

$$\frac{d\Pi_0}{dr} = N\left[1-r\lambda(r)\right]\left(1-F(r)\right)G(r).$$

Note that for $r=0$, $\frac{d\Pi_0}{dr}=0$. Moreover, differentiating again:

$$\begin{aligned}
\frac{d^2\Pi_0}{d^2r} &= N\left[1-\lambda(r)-r\lambda'(r)\right]\left(1-F(r)\right)G(r)-N\left[1-r\lambda(r)\right]f(r)G(r)+. \\
&\quad +N\left[1-r\lambda(r)\right]\left(1-F(r)\right)g(r)
\end{aligned}$$

Note that, as long as $\lambda(r)$ is bounded, for r sufficiently close to 0, $\frac{d^2\Pi_0}{d^2r}>0$: the auctioneer gains from setting a reserve price higher than her valuation. The role of the reserve price is to exclude the bidders with the lowest valuation from the auction. The auctioneer can obtain the same outcome by setting an **entry fee.**

2. Auctions with Interdependent Values and Correlated Bidders' Signals

Setting #2:

- N bidders, submitting their bids simultaneously.
- Each bidder i's private information is summarized by some private information in the form of a signal $X_i \in [0, \omega_i]$.
- The object sold has value V_i to bidder i, where

$$V_i = v_i(X_1, X_2, .., X_N).$$

Where

(1) v_i is non-decreasing in all its variables,

(2) v_i is twice continuously differentiable,

(3) v_i is strictly increasing in X_i,

(4) $v_i(0, 0, .., 0) = 0$ and $E[V_i] < \infty$,

- X_1, X_2, .., X_N could be correlated or independent
- Each bidder i knows the realization x_i of random variable X_i and knows how the values of the other bidders are distributed.
- Bidders have no liquidity or budget constraints.
- All bidders are risk-neutral

With interdependent values the equilibrium bidding strategies adopted in the case of independent values cannot be part of an equilibrium anymore. The reason for that is the **winner's curse**. Too see this effect, let's consider a first price auction. Suppose every bidder follows a symmetric strategies and bids are increasing in valuations. If bidder i wins the auction, her estimate of the value of the object is $E[V|X_i = x_i, Y_i < x_i]$: this is less than $E[V|X_i = x_i]$:

learning that one had won the auction is "bad news" about the value of the object (and this news is worse the more bidders participate in the auction). In equilibrium bidders take this effect into account.

With interdependent value, the revenue equivalence principle does not hold anymore. The "linkage principle" allows to rank the most common forms of auction when values are interdependent. Suppose A is a standard auction in which the highest bid wins the object and suppose that A has a symmetric equilibrium where every bidder bids according to β^A. Consider bidder 1 and assume all other bidders bid according to β^A. Let $W^A(z,x)$ denote the expected price paid by bidder 1 if she wins, she receives signal x, and she bids $\beta^A(z)$. Let $W_2^A(z,x)$ denote the partial derivative of function W^A with respect to its second argument, at (z,x).

Theorem. ***(Revenue Ranking Principle)*** *Let A and B be two standard auctions in which only the winner pays a positive amount. Suppose that each auction has a symmetric and increasing equilibrium such that (1) for all x, $W_2^A(x,x) \geq W_2^B(x,x)$; (2) $W_2^A(0,0) = 0 = W_2^B(0,0)$. Then the expected revenue in A is at least as large as the expected revenue in B.*

This theorem captures the "linkage principle": the greater the linkage between a bidder's own information and the price she expects to pay upon winning, the greater the expected price paid upon winning.

Proof. Consider auction A and suppose that all bidders $j \neq 1$ follow the symmetric equilibrium strategy β^A. The probability that bidder 1 with signal x who bids $\beta^A(z)$ will win is

$$G(z|x) := Prob\left[Y_1 < z | X_1 = x\right].$$

Thus, each bidder chooses z to maximize:

$$\int_0^z v(x,y)g(y|x)dy - G(z|x)W^A(z,x).$$

In equilibrium, it is optimal to choose $z = x$, so the relevant FOC is:

$$v(x,x)g(x|x) - g(x|x)W^A(x,x) - G(x|x)W_1^A(x,x) = 0$$

$$\leftrightarrow$$

$$W_1^A(x,x) = \frac{g(x|x)}{G(x|x)}v(x,x) - \frac{g(x|x)}{G(x|x)}W^A(x,x).$$

Similarly, in auction B:

$$W_1^B(x,x) = \frac{g(x|x)}{G(x|x)}v(x,x) - \frac{g(x|x)}{G(x|x)}W^B(x,x).$$

Thus:

(1) $$W_1^A(x,x) - W_1^B(x,x) = -\frac{g(x|x)}{G(x|x)}\left[W^A(x,x) - W^B(x,x)\right]$$

Define:

$$\Delta(x) := W^A(x,x) - W^B(x,x)$$

so:

(2) $$\Delta'(x) = \left[W_1^A(x,x) - W_1^B(x,x)\right] + \left[W_2^A(x,x) - W_2^B(x,x)\right]$$

Combining (1) and (2):

$$\Delta'(x) = -\frac{g(x|x)}{G(x|x)}\left[W^A(x,x) - W^B(x,x)\right] + \left[W_2^A(x,x) - W_2^B(x,x)\right].$$

By assumption $W_2^A(x,x) - W_2^B(x,x) \geq 0$. Thus, $\Delta(x) \leq 0 \rightarrow \Delta'(x) \geq 0$.

As $\Delta(0) = 0$ it must be the case that $\Delta(x) > 0$ for all $x \in [0, \omega_i]$. QED

The assumption of private values is not crucial for the Revenue Equivalence Theorem: if signals are independently distributed, then in any auction that satisfies the assumptions of the theorem $W^A(z, x)$ does not depend on x, thus $W_2^A(x, x) = 0$, so the revenues in the two auctions are the same by the Revenue Ranking Principle, thus the Revenue Equivalence Theorem holds.