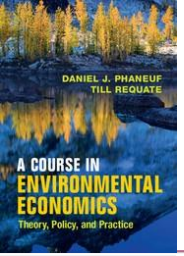


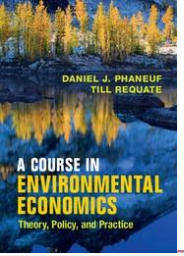
Lecture 7

(based on Chapter 13 in textbook)

Accumulating Pollutants



The Accumulating Pollutant Cost- Damage Model



Introduction

In this chapter we distinguish between

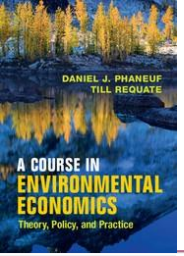
- emissions E_t as a **flow variable**
- and **ambient pollution S_t as a stock**.

- The damage results from the pollution stock (such as the stock of greenhouse gases): $D(S_t)$

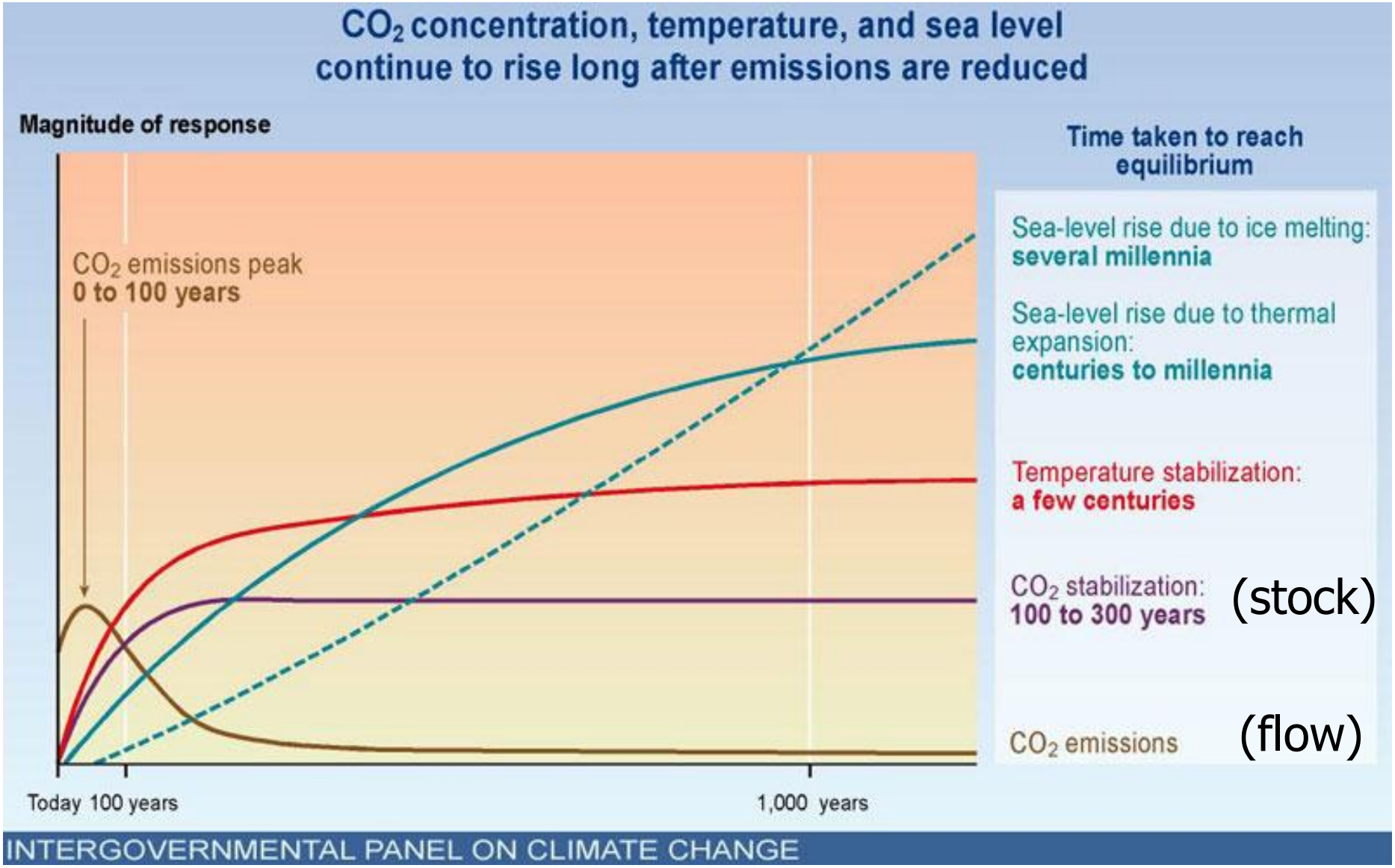
- The abatement cost depend on the flow of emissions: $C(E_t)$

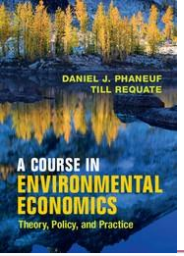
- The (optimal) pollution/abatement problem is therefore a dynamic problem.

- The points of interest are therefore:
  - Characterization of the steady state (a stationary state)
  - The optimal transitional dynamics, depending on the starting point (the stock of pollution we start from)



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13.1.1 Stock Dynamics

- In **discrete time**, the relationship between the emissions E_t and the pollution stock S_t can be written as

$$S_{t+1} = (1-\beta)S_t + E_t \qquad (13.1)$$

- β represents the natural decay rate.

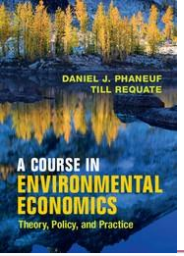
- Rewrite (13.1) as a difference equation according to

$$S_{t+1} - S_t = E_t - \beta S_t \qquad (13.2)$$

- In continuous time, the equation of motion for the pollution stock is the linear differential equation

$$\dot{S}_t = E_t - \beta S_t \qquad (13.3)$$

- β is now the instantaneous decay rate in continuous time.
- $\dot{S}_t = dS_t/dt$



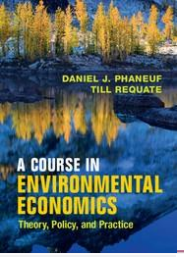
13.1.2 Abatement Cost and Damage

- $C'(E_t) < 0$ and $C''(E_t) > 0$ for $E_t \leq \hat{E}$. (as before in chapter 3)
- $\lim_{E \to 0}\{-C'(E_t\} = \infty$.
- $D'(S_t) > 0$ and $D''(S_t) < 0$.
- The regulator's aim is to minimize the sum of total social costs over time periods running from the present to some endpoint T.
- In discrete time, the social objective function is to select E_t to minimize

$$\sum_{t=0}^{T} \delta^t \left(C(E_t) + D(S_t)\right) \qquad (13.4)$$

s.t. (13.2) and an initial condition $S(0) = S_0$.

- With a finite time horizon there may also be a terminal condition $S(T) = S_T$.

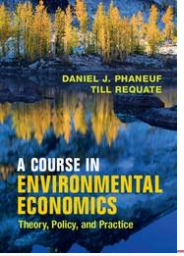


13.1.2 Abatement Cost and Damage

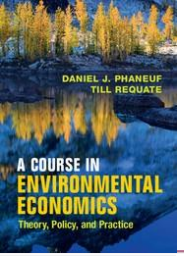
- The discount factor is given by $\delta = 1/(1+r)$.
- The social discount rate is given by $r \geq 0$.
- For $T = \infty$, we assume $\delta < 1$ to guarantee convergence of the sum in (13.4).
- With an infinite time horizon, the social objective function is given by

$$\min_{E_t} \int_0^{\infty} \left(C(E_t) + D(S_t)\right) e^{-rt} dt \qquad (13.5)$$

subject to (13.3), where r is once again the (instantaneous) social discount rate and e^{-rt} is the discount factor converting time *t* costs and benefits into present period values.



Dynamic Optimality



13.2.1 Discrete Time

- In discrete time, we minimize

$$\sum_{t=0}^{T} \delta^t \left(C(E_t) + D(S_t)\right) \tag{13.4}$$

with respect to

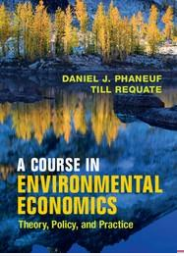
$$S_{t+1} - S_t = E_t - \beta S_t \tag{13.2}$$

- To set up the Lagrange function it is convenient to rewrite (13.2) as

$$\delta^{t-1}\left(S_{t+1} - (1-\beta)S_t - E_t\right) = 0 \tag{13.6}$$

- Write the (present value) Lagrangian as

$$L = \sum_{t=1}^{T} \delta^{t-1}\left(C(E_t) + D(S_t) - \lambda_t \left[S_{t+1} - (1-\beta)S_t - E_t\right]\right) \tag{13.7}$$



13.2.1 Discrete Time

- Differentiating with respect to E_t yields

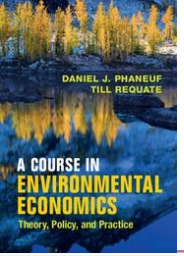
$$C'(E_t) + \lambda_t = 0 \qquad (13.8)$$

or

$$-C'(E_t) = \lambda_t \qquad (13.9)$$

- It follows that in each period, the current marginal abatement cost should be equal to **shadow cost of the current period's dynamic constraint**, which is the (shadow) social cost of the future pollution stock.

- The shadow price can be interpreted as the optimal tax (or price for tradable permits),
- In a decentralized world, the marginal abatement cost must be equal to the optimal tax rate.



13.2.1 Discrete Time

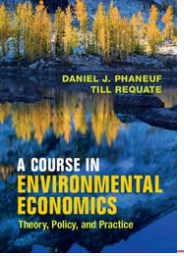
- Differentiate the Lagrangian (13.7) with respect to the pollution stock S_t to obtain

$$D'(S_t) - \frac{1}{\delta}\lambda_{t-1} + \lambda_t(1-\beta) = 0 \qquad (13.10)$$

- By using $\delta = 1/(1+r)$, and moving forward one time unit, it yields

$$D'(S_{t+1}) + \lambda_{t+1}(1-\beta) = (1+r)\lambda_t \qquad (13.11)$$

- Along the optimal emission path, increasing emissions today by one unit, and saving the avoided marginal abatement cost, defined by $\lambda_t = -C'(E_t)$, yields a payoff of $(1+r)\lambda_t$ tomorrow.



13.2.1 Discrete Time

- In an infinite time horizon the system must converge to a steady state, with $\lim_{t\to\infty} E_t = E^*$, $\lim_{t\to\infty} S_t = S^*$ and $\lim_{t\to\infty} \lambda_t = \lambda^*$.

- Using the steady state values in

$$D'(S_{t+1}) + \lambda_{t+1}(1-\beta) = (1+r)\lambda_t \quad \Rightarrow \quad D'(S^*) + \lambda^*(1-\beta) = (1+r)\lambda^* \qquad (13.11)$$

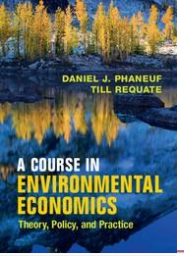
and substituting using

$$-C'(E_t) = \lambda_t \qquad (13.9)$$

it yields

$$-C'(E^*) = \frac{D'(S^*)}{r+\beta} \qquad (13.12)$$

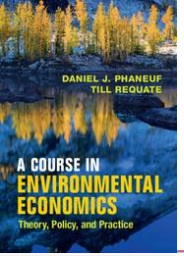
- The marginal abatement cost is equal to the discounted sum of marginal damages.



13.2.2 Continuous Time

- The Hamiltonian is the sum of the current objective (social cost in our context), plus the marginal future cost or benefit of increasing the stock variable by a marginal amount.
- The current value Hamiltonian assesses the instantaneous objective at some point in time t, as if we are starting from t=0.
- The present value Hamiltonian assesses the objective at some later point in time t from the present perspective, and therefore the objective is multiplied by the discount factor e^{-rt}.
- The current value Hamiltonian is given by

$$H(E_t, S_t, \lambda_t) = \left[C(E_t) + D(S_t)\right] + \lambda_t \left[E_t - \beta S_t\right] \qquad (13.13)$$



13.2.2 Continuous Time

- In continuous time we need the so-called Hamiltonian

$$H(E,S,\lambda) = C(E) + D(S) + \lambda[E - \beta S] \qquad (13.14)$$

- The F.O.C's for the optimal emission path are characterized by

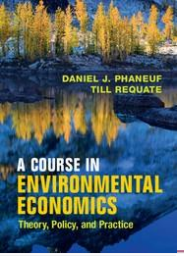
$$\frac{\partial H}{\partial E} = 0 \quad \rightarrow \quad -C'(E) = \lambda \qquad (13.15)$$

$$-\frac{\partial H}{\partial S} = \dot{\lambda} - r\lambda \quad \rightarrow \quad D'(S) - \lambda\beta - \dot{\lambda} = r\lambda \qquad (13.16)$$

and the transversality (or terminal) condition

$$\lim_{t\to T} e^{-rt}\lambda_t S_t = 0 \qquad (13.17)$$

which is equivalent to $\lim_{t\to\infty} e^{-rt}\lambda_t = 0$ in a steady state with infinite horizon.

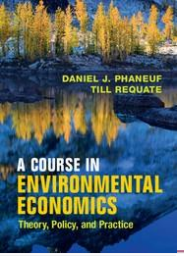


13.2.3 The Steady State

- In the steady state, the variables converge to constants over time: $\dot{E} = 0, \dot{S} = 0, \dot{\lambda} = 0$.
- Setting $\dot{S} = 0$ in (13.3), it follows: $E^* = \beta S^*$.
- Setting $\dot{\lambda} = 0$ in (13.16), using (13.15), and rearranging it follows:

$$-C'(E^*) = \lambda = \frac{D'(S^*)}{r + \beta} \qquad (13.18)$$

- At the optimal steady state, the marginal abatement cost is equal to the present value of all future damages arising by one more unit of emissions.
- The discount rate for emissions is then the sum of the discount rate for costs and benefits, plus the natural decay rate.



13.2.3 The Steady State

Comparative Statics for the Steady State

- Rewrite (13.18) as

$$-C'(E^*)(r+\beta) = D'(S^*) \tag{13.19}$$

and differentiate the equation system consisting of $E^* = \beta S^*$ and (13.19) with respect to r, yielding

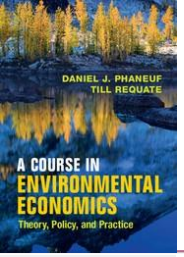
$$\frac{\partial E^*}{\partial r} = \beta \frac{\partial S^*}{\partial r} \tag{13.20}$$

$$-C''(E^*)(r+\beta)\frac{\partial E^*}{\partial r} - C'(E^*) = D''(S^*)\frac{\partial S^*}{\partial r}$$

- Using Cramer's rule yields

$$\frac{\partial E^*}{\partial r} = \frac{-\beta C'(E^*)}{D''(S^*) + \beta(r+\beta)C''(E^*)} > 0 \tag{13.21}$$

$$\frac{\partial S^*}{\partial r} = \frac{-C'(E^*)}{D''(S^*) + \beta(r+\beta)C''(E^*)} > 0$$



13.2.3 The Steady State

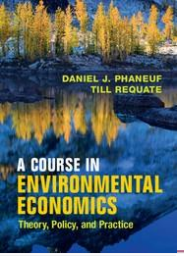
- Doing the same exercise with respect to the decay rate, it yields

$$\frac{\partial E^*}{\partial \beta} = \frac{-\beta C'(E^*) + S^* \cdot D''(S^*)}{D''(S^*) + \beta(r+\beta)C''(E^*)} > 0 \qquad (13.22)$$

$$\frac{\partial S^*}{\partial \beta} = \frac{-C'(E^*) - S^* \cdot (r+\beta) \cdot C''(S^*)}{D''(S^*) + \beta(r+\beta)C''(E^*)}$$

- $\partial E^*/\partial \beta$ is unambiguously positive – a faster decay allows for higher emissions.

- $\partial S^*/\partial \beta$ is ambiguous as two effects can be observed: higher emissions contribute to a higher pollution stock, whereas a faster decay lets the pollution stock shrink.

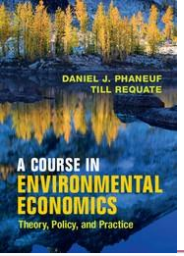


13.2.3 The Steady State

- Substitute S* into the numerator of the second equation in (13.22) using $S^* = E^*/\beta$, it yields

$$\frac{\partial S^*}{\partial \beta} > 0 \quad \Leftrightarrow \quad \frac{-C'(E^*)}{E^* \cdot C''(E^*)} > \frac{r+\beta}{\beta} \tag{13.23}$$

- The pollution stock increases with the decay rate, if and only if the elasticity of the (aggregate) marginal abatement cost curve is sufficiently large.



13.2.4 Optimal Path to the Steady State

- Differentiate (13.15) with respect to time to obtain

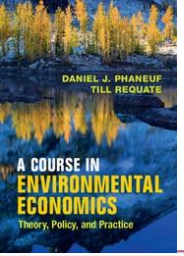
$$-C''(E)\cdot\dot{E}=\dot{\lambda} \tag{13.24}$$

- Using (13.15) and (13.24) to eliminate λ and $\dot{\lambda}$ from (13.16), yielding the equation of motion for emissions as

$$\dot{E}=\frac{D'(S)+(r+\beta)C'(E)}{C''(E)} \tag{13.25}$$

$$\dot{E}=\frac{\frac{D'(S)}{r+\beta}-(-C'(E))}{\frac{C''(E)}{r+\beta}} \tag{13.26}$$

- The evolution of emissions is driven by the difference between the present discounted value of marginal damage, and marginal abatement cost.



13.2.5 Phase Diagrams

Phase Diagram in Control/State Space

- The control/state diagram can be constructed by using equations

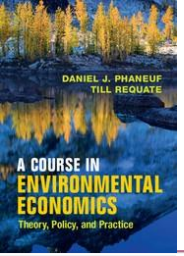
$$\dot{E} = \frac{D'(S) + (r + \beta)C'(E)}{C''(E)} \tag{13.25}$$

$$\dot{S}_t = E_t - \beta S_t \tag{13.3}$$

to obtain

$$\dot{S} = E - \beta S$$

$$\dot{E} = \frac{D'(S) + (r + \beta)C'(E)}{C''(E)} \tag{13.27}$$



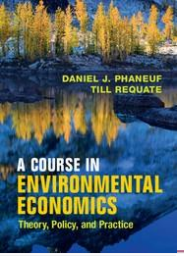
13.2.5 Phase Diagrams

- The $\dot{S} = 0$ curve (also called isocline), which is driven by the natural decay process in (13.3), can be written as $E_N(S) = \beta S$.

- The control equation for $\dot{E} = 0$ can be written as

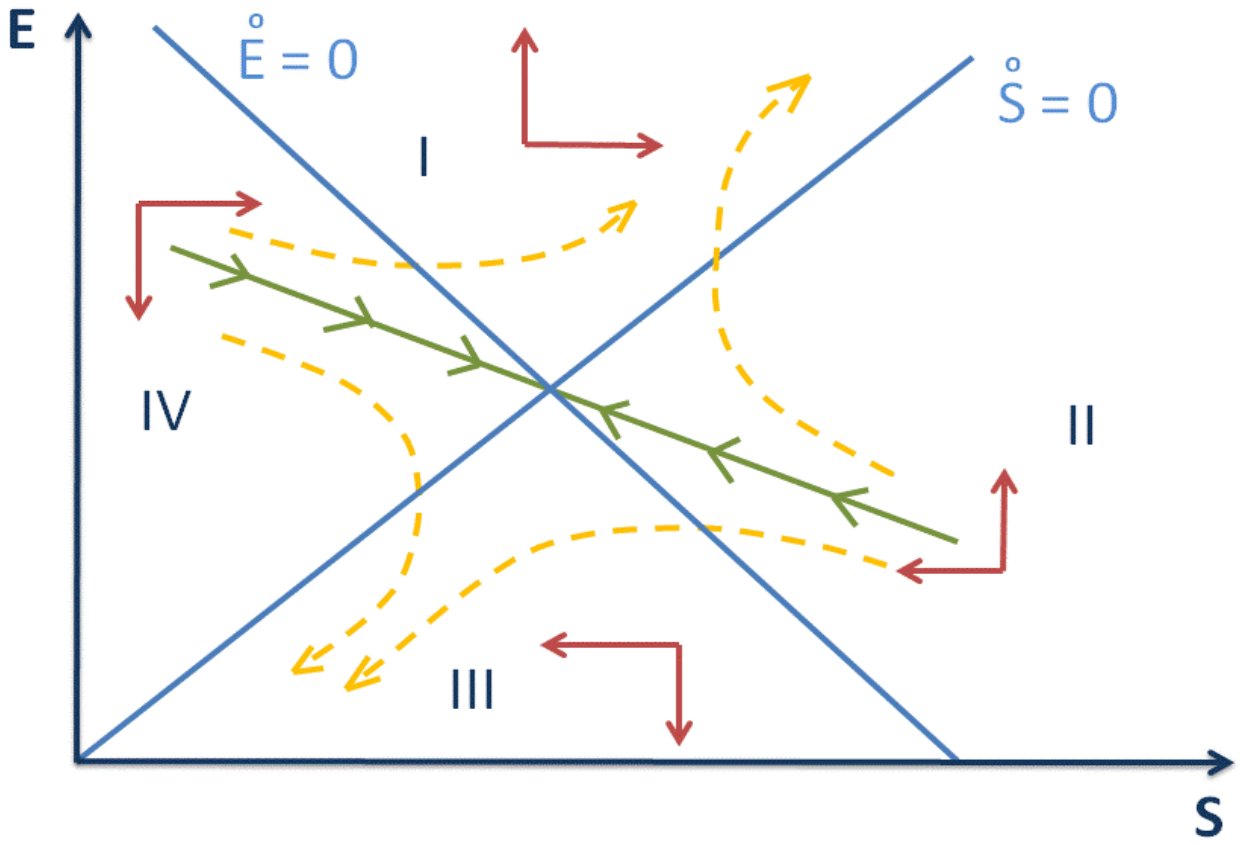
$$E_C(S) = C_E^{-1}\left(-D'(S)/(r+\beta)\right) \qquad (13.28)$$

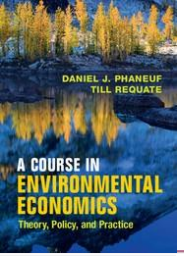
- The intersection point of the two curves defines the steady state of the dynamic system.



13.2.5 Phase Diagrams

Figure 13.1a The phase diagram in the S/E- space





13.2.5 Phase Diagrams

Phase Diagram in State/Co-state Space

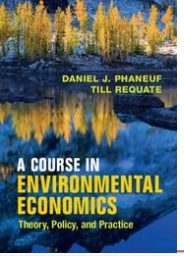
- The co-state variable can be interpreted as the shadow price for emissions, and so is also the optimal emission tax (or the optimal market price for tradable emission permits) in a regulated competitive economy.

- Rearranging

$$-\frac{\partial H}{\partial S} = \dot{\lambda} - r\lambda \quad \rightarrow \quad D'(S) - \lambda\beta - \dot{\lambda} = r\lambda \tag{13.16}$$

yields:

$$\dot{\lambda} = D'(S) - \lambda(r + \beta) \tag{13.29}$$



13.2.5 Phase Diagrams

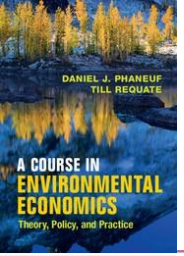
$$\dot{S} = C_E^{-1}(-\lambda) - \beta S \tag{13.30}$$

- From (13.29) and (13.30), respectively, it can be seen that the $\dot{\lambda} = 0$ curve is

$$\lambda = \frac{D'(S)}{r + \beta} \tag{13.31}$$

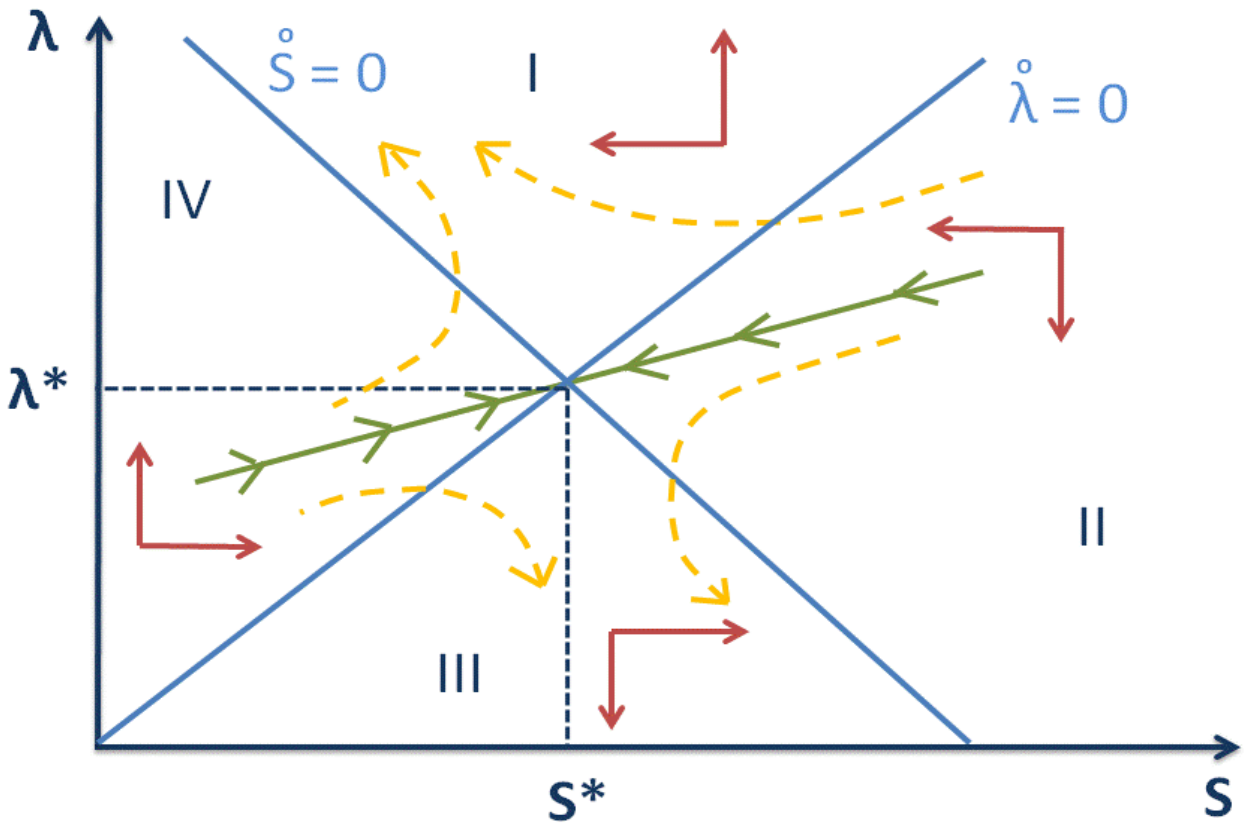
and the $\dot{S} = 0$ curve is

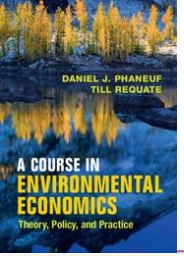
$$\lambda = -C'(\beta S) \tag{13.32}$$



13.2.5 Phase Diagrams

Figure 13.1b The phase diagram in the S/ λ-space





13.2.5 Phase Diagrams

Optimal Feedback Strategy

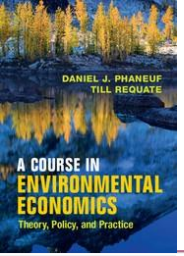
- Since the dynamic process under consideration is an autonomous process, which does not depend on time directly, we can further use the optimality conditions (13.3), (13.15) and (13.16) to characterize the optimal feedback strategy $E(S)$.

- Differentiate

$$\frac{\partial H}{\partial E} = 0 \quad \rightarrow \quad -C'(E) = \lambda \tag{13.15}$$

with respect to time to obtain

$$-C''(E) \cdot E'(S) \cdot \dot{S} = \dot{\lambda} \tag{13.33}$$



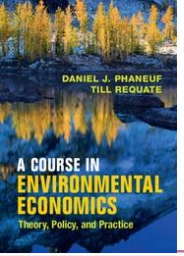
13.2.5 Phase Diagrams

- A differential equation for the feedback strategy $E(S)$ can be obtained as

$$-D'(S)-(\beta+r)C'(E)+C''(E)E'(S)(E-\beta S)=0 \qquad (13.34)$$

or

$$E'(S)=\frac{D'(S)+(\beta+r)C'(E)}{C''(E)\left(E-\beta S\right)} \qquad (13.35)$$



Example 13.1

- The results for quadratic abatement cost and damage functions can be illustrated as

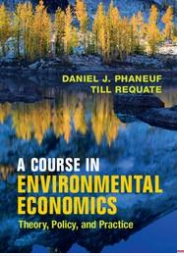
$$C(E) = \frac{(a - bE)^2}{2b} \tag{13.36}$$

$$D(S) = d\frac{S^2}{2},$$

for which the steady state stock is straightforward to calculate as

$$S^* = \frac{(r + \beta)a}{d + (r + \beta)b\beta} \tag{13.37}$$

- From this the steady emissions level can be determined as $E^* = \beta S^*$ and the co-state variable steady state level as $\lambda^* = a/b - E^*$.



Example 13.1

- For the linear marginal abatement cost and marginal damage curves the optimal emission, pollution and co-state variable paths can be presented as a function of time in closed form:

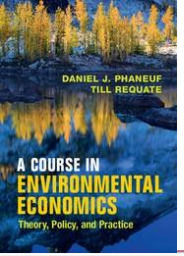
$$S(t) = S^* + (S_0 - S^*) \cdot K_S \cdot e^{\mu t} \qquad (13.38)$$

$$E(t) = E^* - (S_0 - S^*) \cdot K_E \cdot e^{\mu t} \qquad (13.39)$$

$$\lambda(t) = \lambda^* + (S_0 - S^*) \cdot K_\lambda \cdot e^{\mu t} \qquad (13.40)$$

- From these expressions can be seen that all variables converge to the steady state as t goes to infinity.

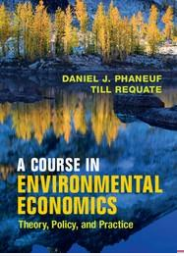
- Note also that if the initial stock exceeds the stead state level so that $S_0 > S^*$ the path $S(t)$ approaches S^* from above, and in the opposite case from below.



Example 13.1

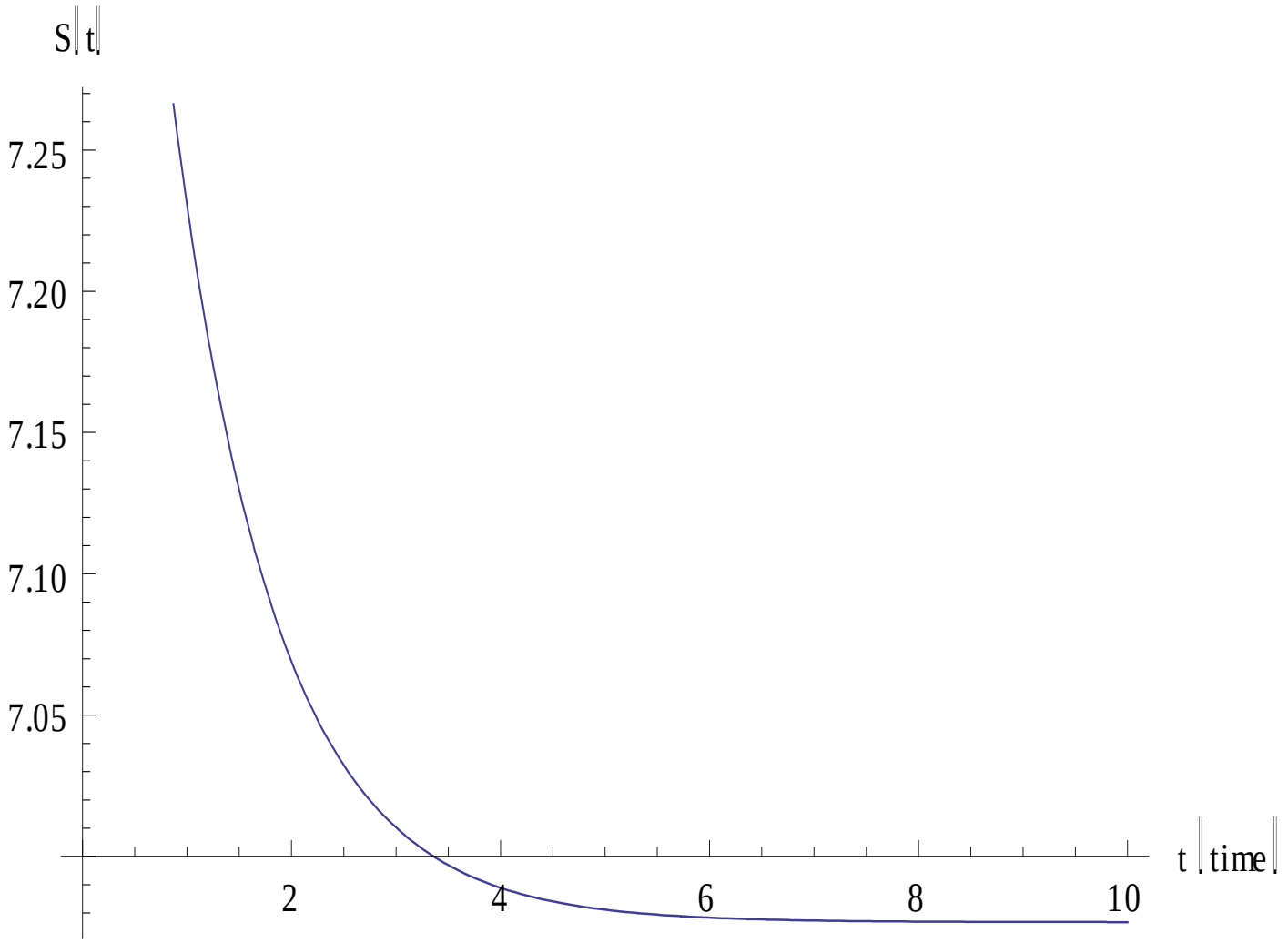
- For specific parameter values, the optimal time paths for the pollution stock, emissions, and the co-state variable are displayed in

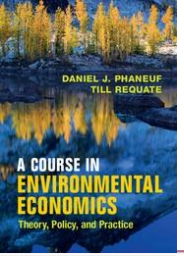
- Figures 13.2a to 13.2c for the case of an initial pollution stock exceeding the steady state, and in Figures 13.2d to 13.2f for the case of an initially low pollution stock.



Example 13.1

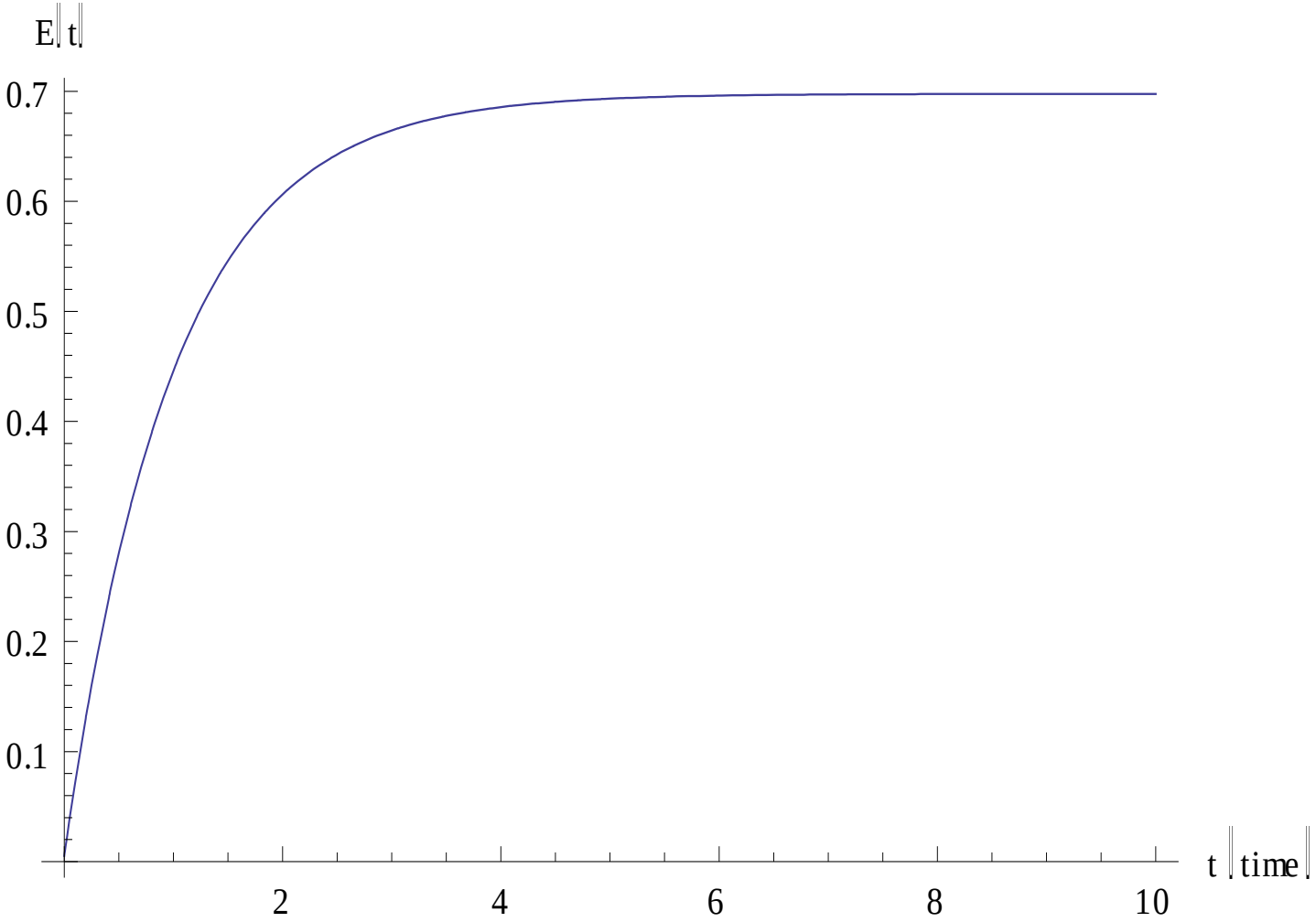
Figure 13.2a: The optimal time path of the pollution stock, starting with an initial pollution stock exceeding the steady-state value.

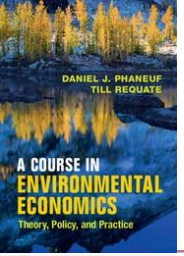




Example 13.1

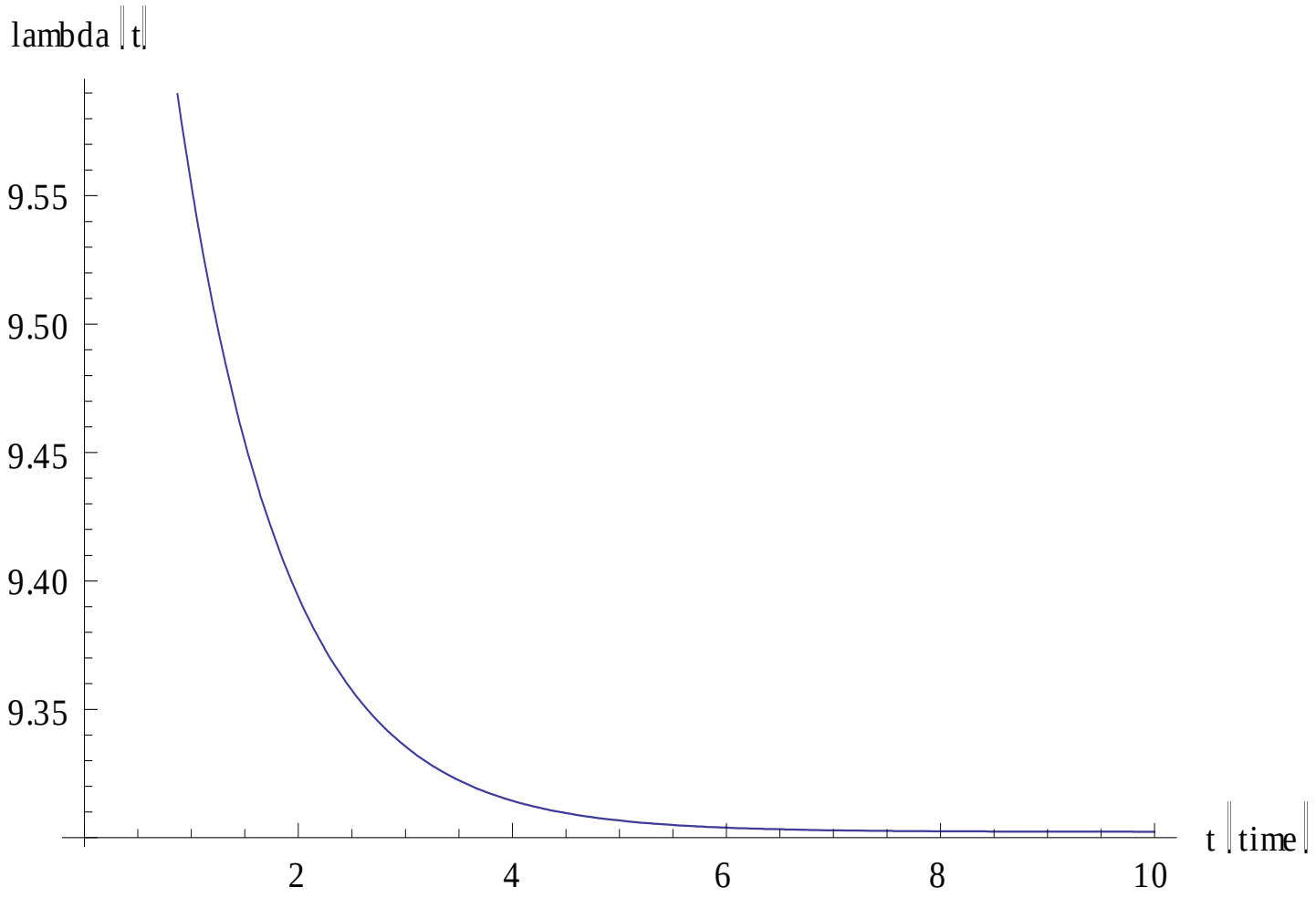
Figure 13.2b: The optimal time path of emissions (with an initial pollution stock exceeding the steady-state value).

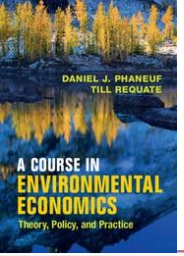




Example 13.1

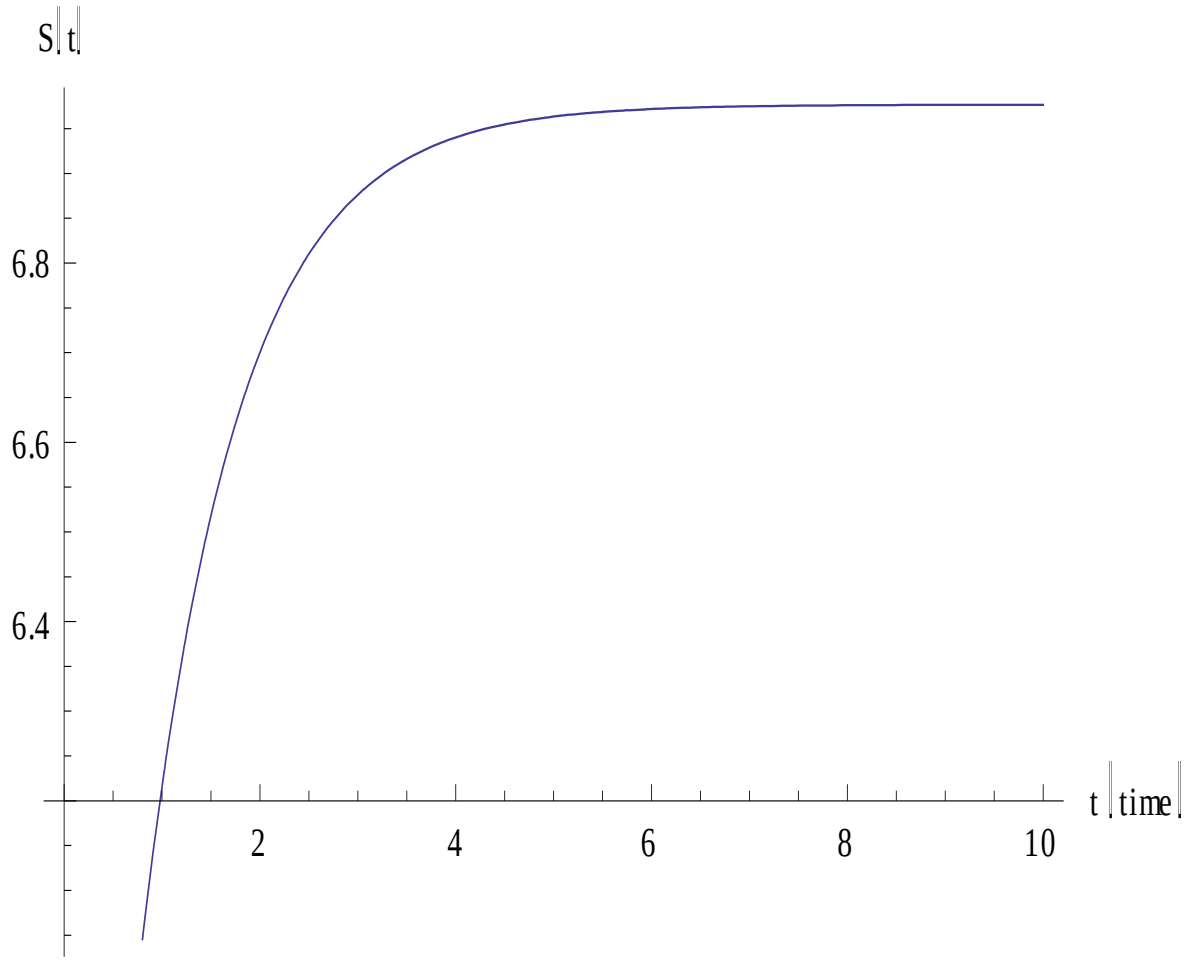
Figure 13.2c: The optimal time path of the co-state variable (the Pigouvian tax rate) over time (with an initial pollution stock exceeding the steady-state value).

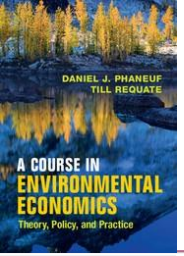




Example 13.1

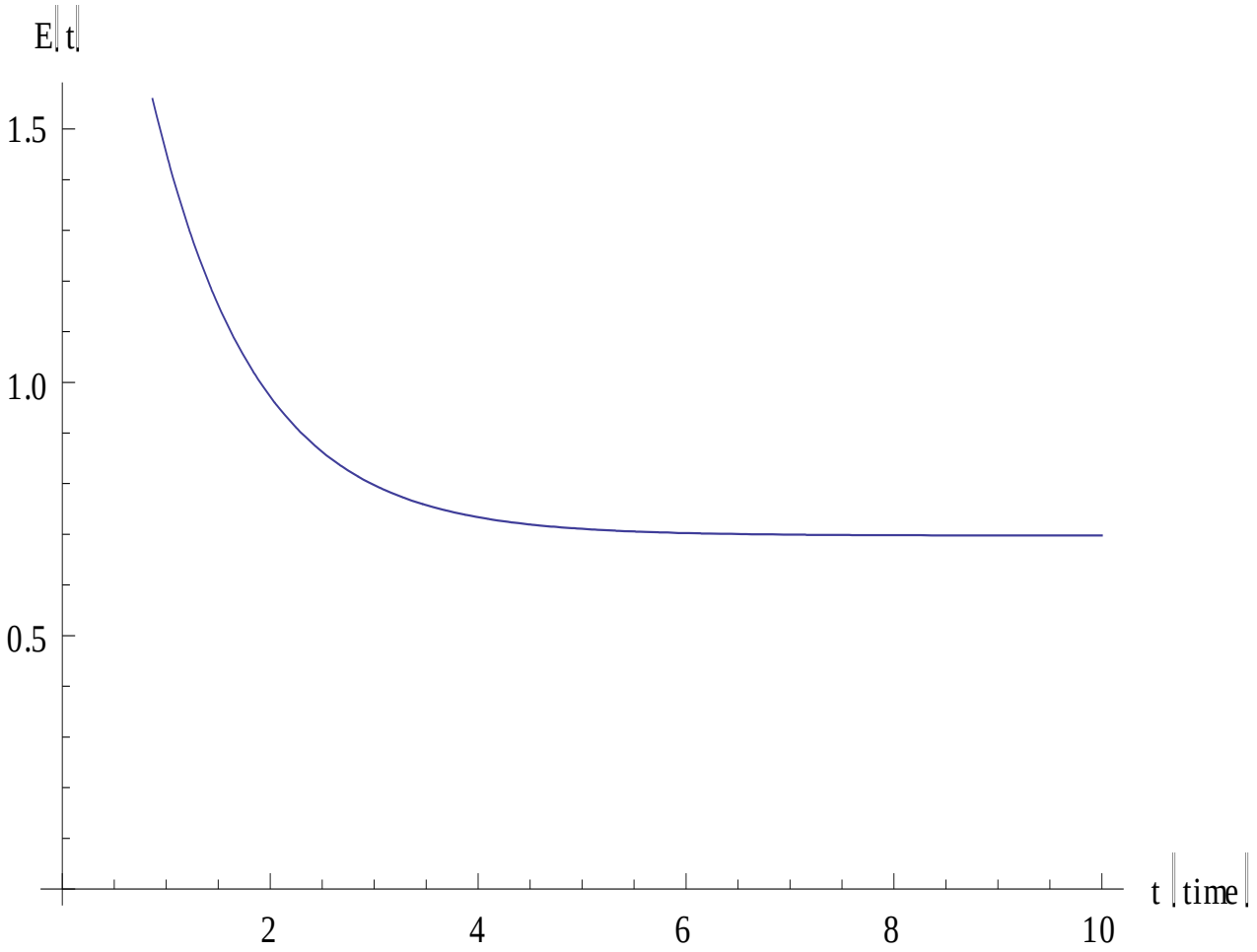
Figure 13.2d: The optimal evolution of the pollution stock over time, starting with an initial pollution stock below the steady-state value.

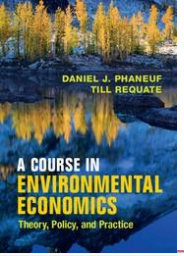




Example 13.1

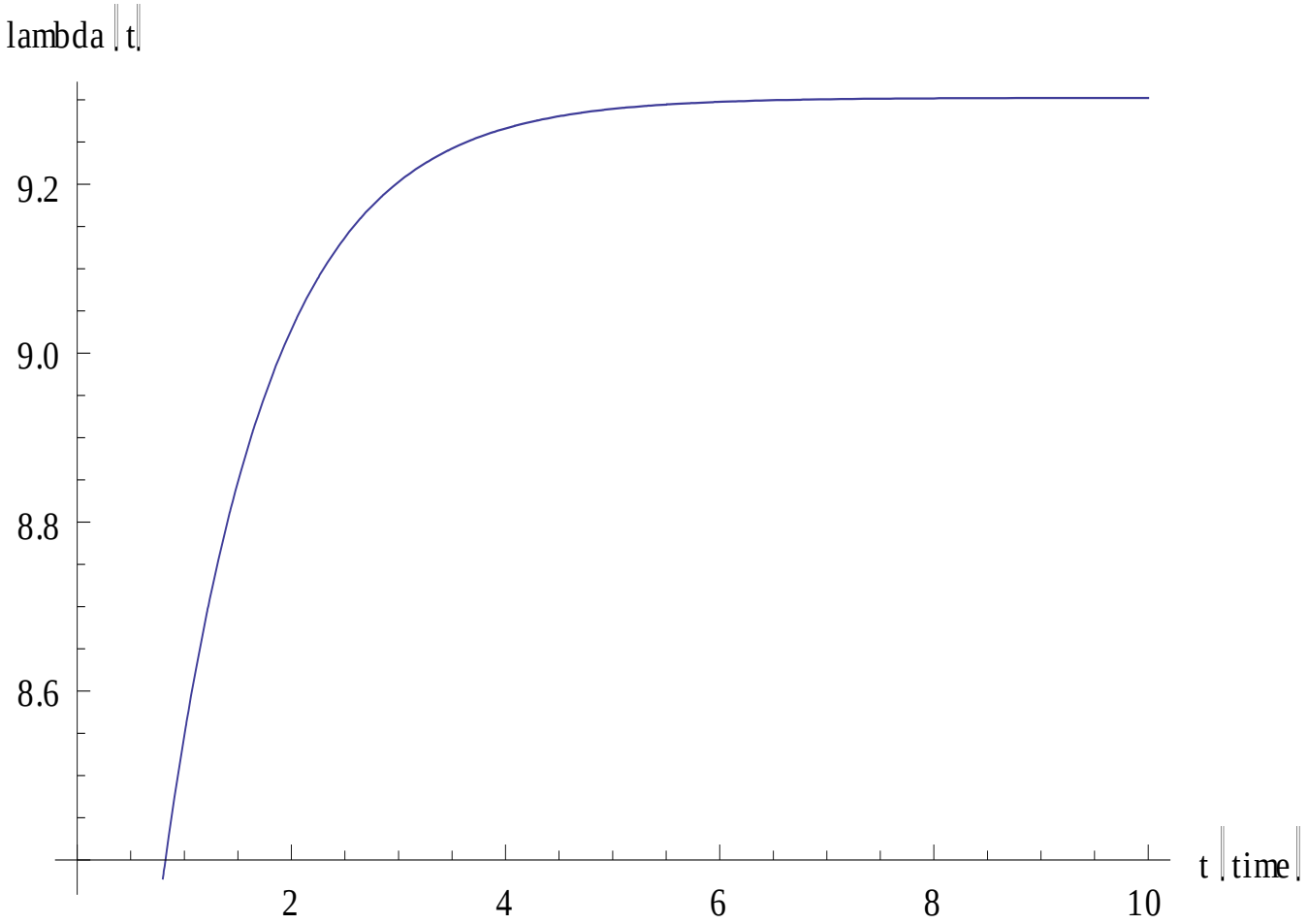
Figure 13.2e: The optimal evolution of emissions over time (with an initial pollution stock below the steady-state value).

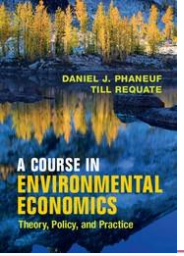




Example 13.1

Figure 13.2f: The optimal evolution of co-state variable (the Pigouvian tax rate) over time (with an initial pollution stock below the steady-state value).





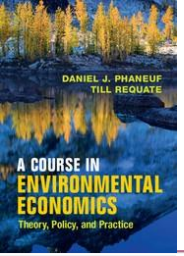
Example 13.1

- Focusing on the optimal feedback strategy, the functional forms, (13.35) becomes

$$E'(S) = \frac{d \cdot S - (\beta + r)(a - bE)}{b(E - \beta S)} \tag{13.41}$$

- Solving this equation, it follows the simple feedback rule, where optimal emissions are a linear function of the pollution stock:

$$E(S) = \frac{a}{b} - \frac{d}{b(\beta + r)} S \tag{13.42}$$



13.2.6 Policy Instruments and Decentralized Decision-Making

- In the case of emission taxes, a representative firm's total cost is given by

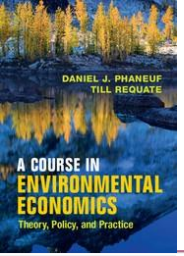
$$TC(E) = C(E) + \tau E \tag{13.43}$$

- If the regulator sets a tax path $\{\tau_t\}_{t\geq 0}$, the firm's objective function over the total planning horizon is

$$\min_{E_t} \int_0^{\infty} \left(C(E_t) + \tau_t E_t\right) dt \tag{13.44}$$

- Since the firm's profit or cost does not depend on the pollution stock, it is sufficient to minimize its current cost at each point in time. This yields

$$-C'(E_t) = \tau_t \tag{13.45}$$

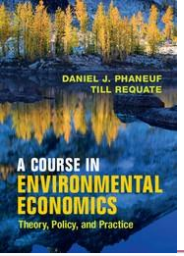


13.2.6 Policy Instruments and Decentralized Decision-Making

- Since the first-order condition for the social optimum requires $-C'(E_t) = \lambda_t$, the regulator has to set $\tau_t = \lambda_t$.

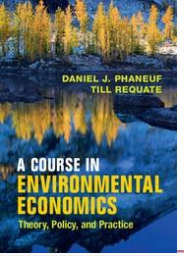
- Under a regime of tradable emission permits, the optimal amount of permits has to be set equal to the optimal level of emissions, resulting in $L_t = E_t^*$.

- If permit markets are competitive, the equilibrium permit price must be equal to the optimal shadow price of emissions, so that $\sigma_t = \lambda_t$.



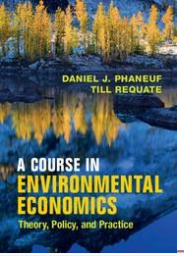
Exhaustible Resources and Pollution Control

- probably to be skipped -



13.3.1 Pure Resource Extraction Model

- Consider a model in which a representative household consumes a small fraction of some finite, exhaustible resource at each point in time.
- The total consumer utility is given by $U(x_t, z_t) = u(x_t) + z_t$.
- $c \geq 0$.
- Let R_t be the stock of the remaining resource stock in situ, where $R_0 = R(0)$ is the initial stock at point in time t=0.



13.3.1 Pure Resource Extraction Model

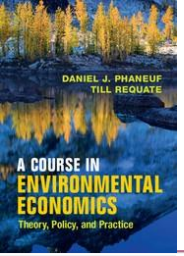
- To characterize the socially optimal extraction of the resource, it is assumed that the representative household owns the resource stock, and consider its utility maximization problem

$$\max_{x=\{x_t\}} \int_0^{\infty} \left(u(x) + y - cx\right) e^{-rt} dt \qquad (13.46)$$

subject to

$$\dot{R} = -x \qquad (13.47)$$

and the initial condition $R_0 = R(0)$ where r is the social discount rate and $z = y - cx$ is consumption of the numeraire.



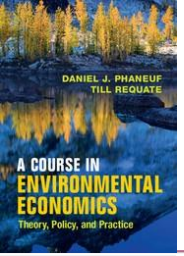
13.3.1 Pure Resource Extraction Model

- The equation of motion (13.47) shows that the resource stock in situ (i.e. in the ground) diminishes at each point in time, based on the extraction rate per unit of time, and there is no renewal of the stock.

- To solve this problem, we set up the current value Hamiltonian, omitting time indices, as

$$H(x,R,\varphi) = u(x) - cx - \varphi x \qquad (13.48)$$

where φ is the co-state variable with the respect to resource stock, and we have dropped the constant y.



13.3.1 Pure Resource Extraction Model

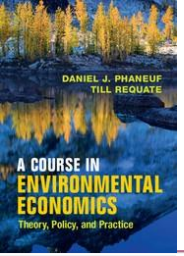
- The F.O.C's are then

$$\frac{\partial H}{\partial x} = u'(x) - c - \varphi = 0 \quad \rightarrow \quad u'(x) = c + \varphi \tag{13.49}$$

$$-\frac{\partial H}{\partial R} = 0 = \dot{\varphi} - r\varphi \quad \rightarrow \quad \dot{\varphi} = r\varphi \tag{13.50}$$

$$\lim_{t\to\infty} \varphi e^{-\delta t} = 0 \tag{13.51}$$

- From (13.50), it follows that the co-state variable, which is the shadow value of the resource, must increase over time; by (13.49) this is likewise the case for the marginal utility of consuming the resource.



13.3.1 Pure Resource Extraction Model

The Competitive Market Solution

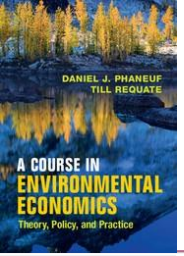
- For simplicity assume that τ represents both the social discount rate and the market interest rate.
- Assume that a representative firm owns the resource stock, but behaves as a price taker.
- Note that this means that the firm takes the whole time path of the price as given. The representative firm solves

$$\max_{x=\{x_t\}} \int_0^{\infty} (p-c)\cdot x \cdot e^{-rt} dt \qquad (13.52)$$

s.t

$$\dot{R} = -x \qquad (13.53)$$

and once again the initial condition $R_0 = R(0)$.



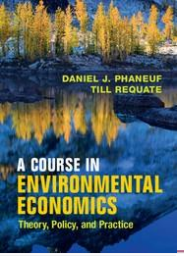
13.3.1 Pure Resource Extraction Model

- The current value Hamiltonian for this problem is now

$$H(x,z,\varphi) = (p-c)x - \varphi x \tag{13.54}$$

and the derivative with respect to the control variable is $\frac{\partial H}{\partial x} = p - c - \varphi$, which is independent of x.

- The equilibrium price is given by $p = c - \varphi$.



13.3.1 Pure Resource Extraction Model

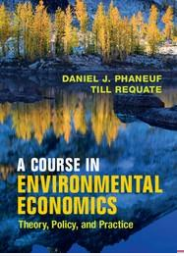
- The additional F.O.C's are the same as the utility maximization case, provided that the market interest rate coincides with the social discount rate

$$\dot{\varphi} = r\varphi \qquad (13.55)$$

$$\lim_{t\to\infty} \varphi_t e^{-rt} = 0$$

- The utility maximizing consumer sets $u'(x) = p$.

- It yields that the market price for the extracted resource consists of two parts: the marginal cost of extraction, and a part that is known as the resource royalty, which can also be interpreted as a scarcity rent.



13.3.1 Pure Resource Extraction Model

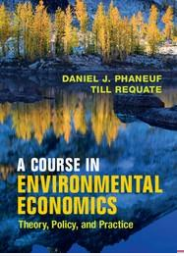
- If by assumption the markets have perfect foresight such that the resource is used up completely, meaning that the initial price can be uniquely determined.

$$\int_0^{\infty} x_t(p_t)dt = R_0 \qquad (13.56)$$

- From the equality $u'(x) = p = c - \varphi$, it follows that decentralized market solution is socially optimal.

- The royalty part of the resource price increases exponentially, and thus slows down exploitation as the resource gets scarcer.

- From this it follows that scarcity of the resource is not a motivation for a government to correct for overexploitation, if property rights are well defined.



13.3.2 Optimal Extraction of a Polluting Resource

- Identify the flow of extraction with emissions, so that $E = x$. This implies that instead of (13.46), our objective function is now

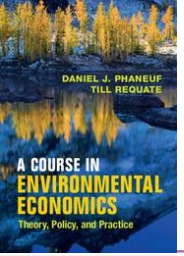
$$\max_{x=\{x_t\}} \int_0^{\infty} \left(u(x) + y - cx - D(S)\right) e^{-rt} dt \qquad (13.57)$$

s.t. the dynamics of resource depletion

$$\dot{R} = -x \qquad (13.58)$$

and the dynamics of emission accumulation

$$\dot{S} = x - \beta S \qquad (13.59)$$



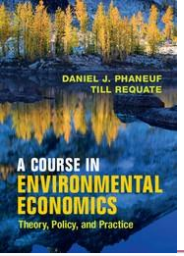
Proposition 13.1

Under the conditions of the model:

a) If the initial stock of pollution S_0 is sufficiently low, both the pollution stock and the co-state variable – i.e. the instantaneous shadow price – are inverted U-shaped.

b) If the initial stock of pollution S_0 is sufficiently high, both the pollution stock and the co-state variable are decreasing along the optimal extraction/emission path.

c) If $D'''(\cdot) \geq 0$, and the initial stock of pollution S_0 is sufficiently low (high), the optimal extraction path for x converges monotonically towards zero (is inverted U-shaped at the beginning and then eventually converges to zero).



13.3.2 Optimal Extraction of a Polluting Resource

Figure 13.3a: Optimal time path of the pollution stock, starting with an initially low pollution stock

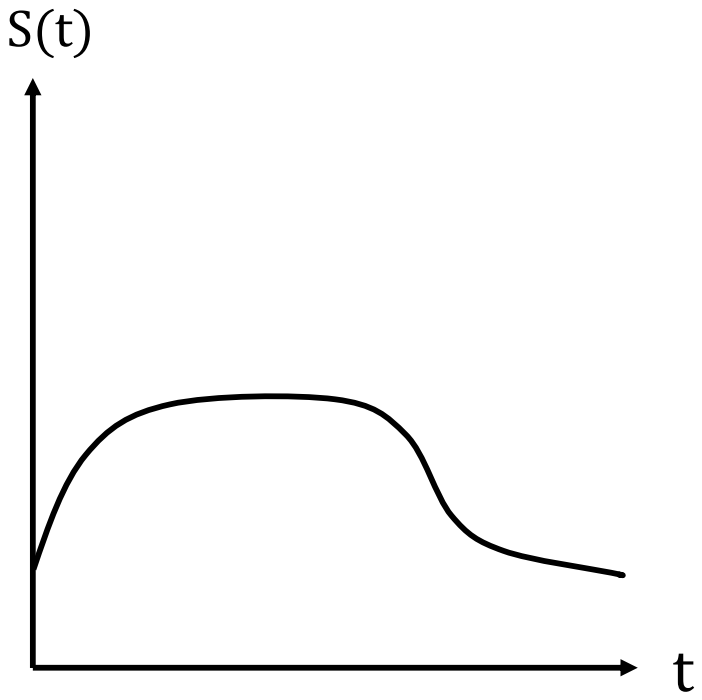
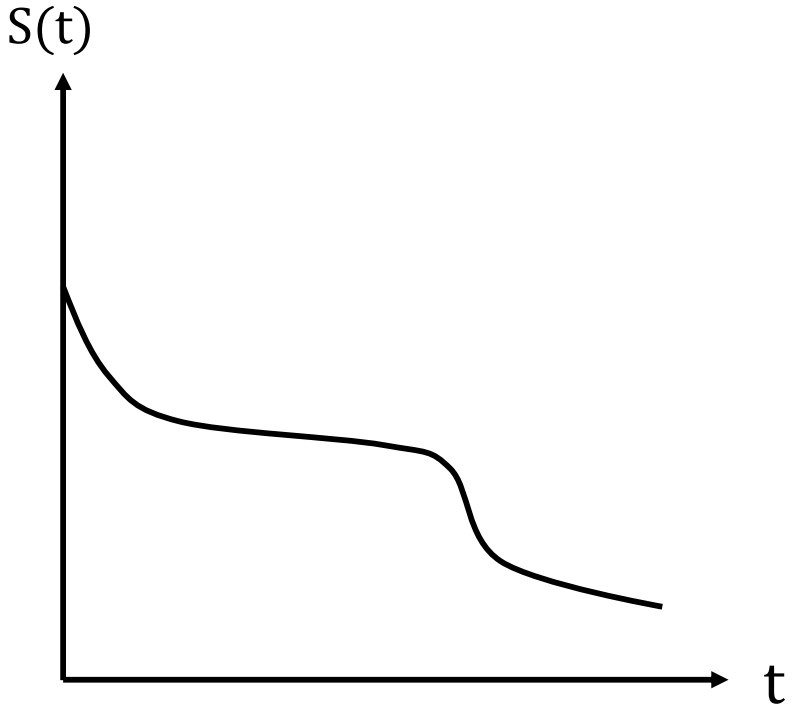
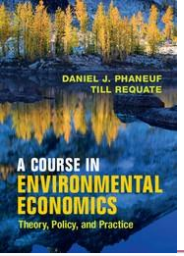


Figure 13.3b: Optimal time path of the pollution stock, starting with an initially high pollution stock.





13.3.2 Optimal Extraction of a Polluting Resource

Figure 13.3c: Optimal time path of the pollution stock's shadow price starting with an initially low pollution stock.

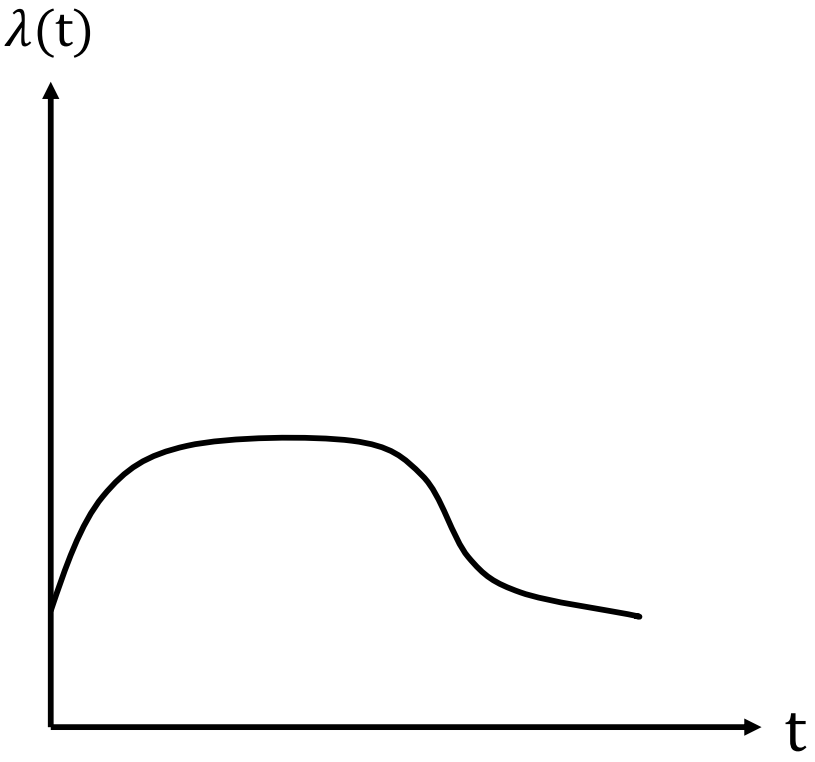
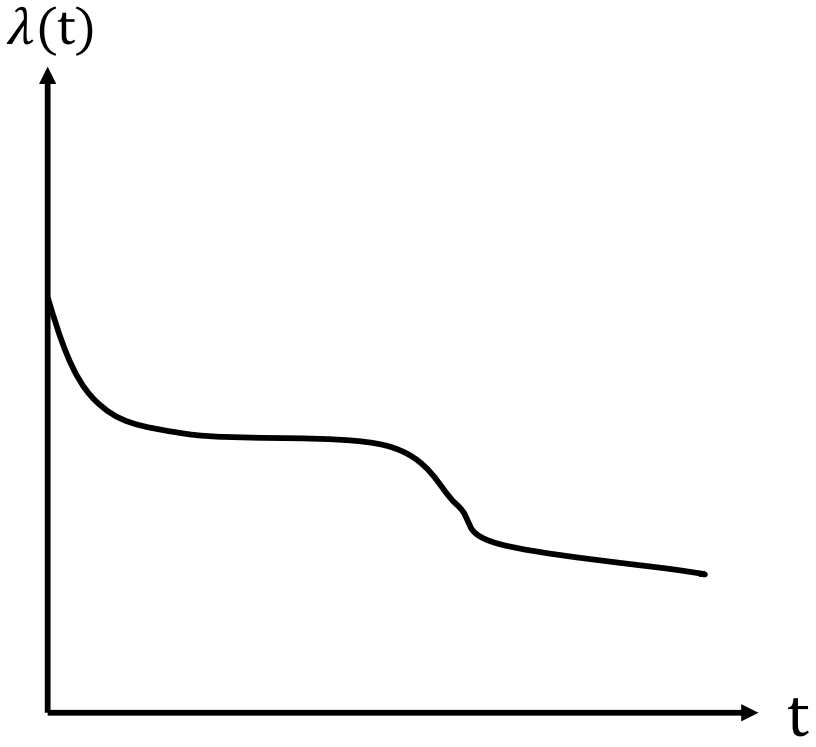
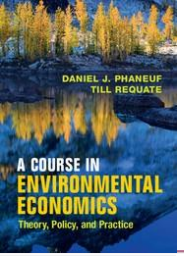


Figure 13.3d: Optimal time path of the pollution stock's shadow price starting with an initially high pollution stock.





13.3.2 Optimal Extraction of a Polluting Resource

Figure 13.3e: The optimal extraction path over time, starting with an initially low pollution stock.

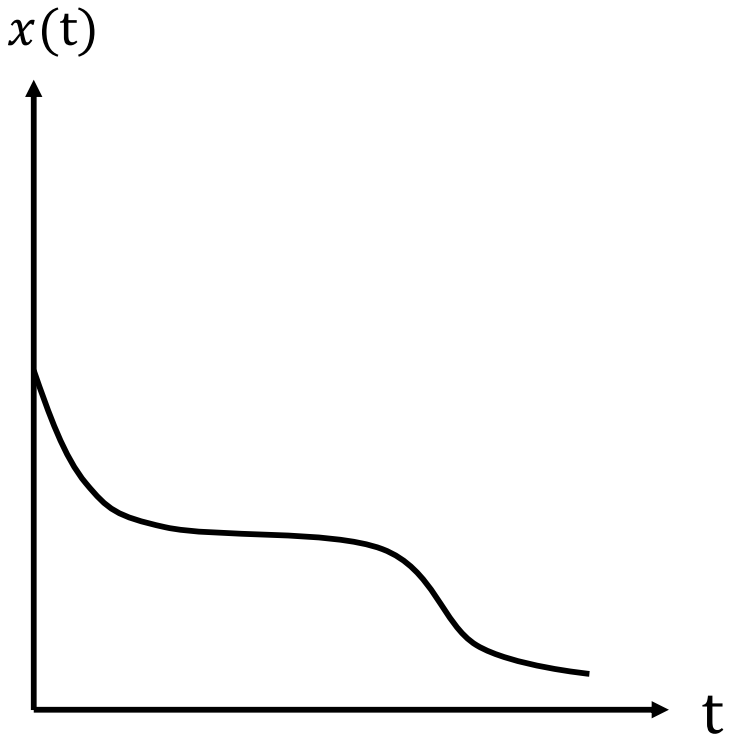
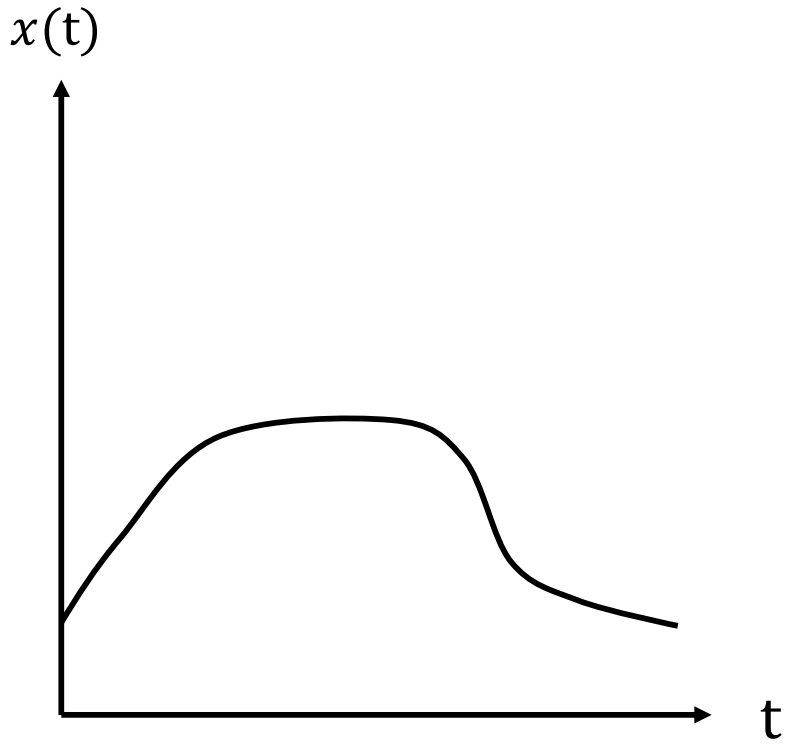
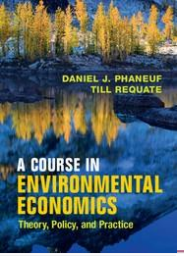


Figure 13.3f: The optimal extraction path over time, starting with an initially high pollution stock.





13.3.2 Optimal Extraction of a Polluting Resource

Figure 13.3g: The optimal time path of the remaining resource stock (all cases).

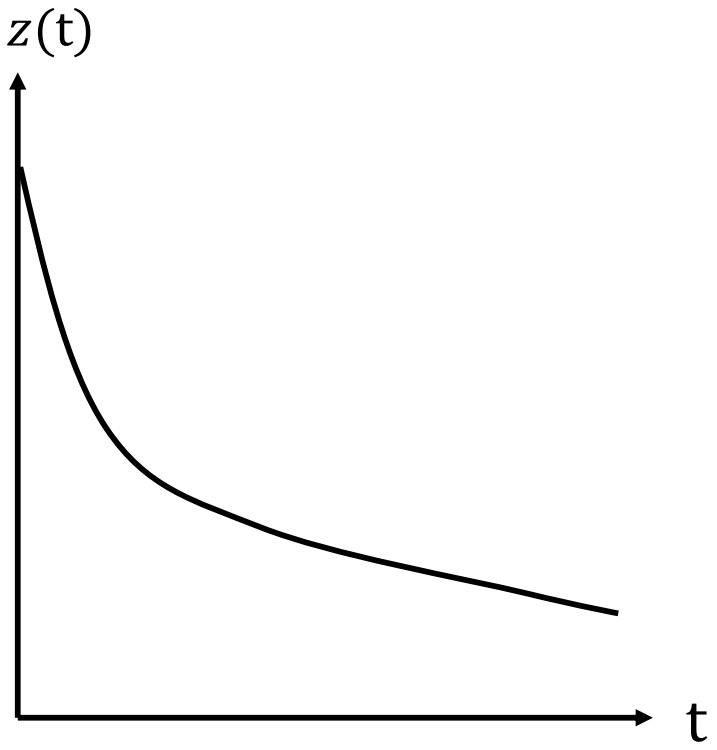
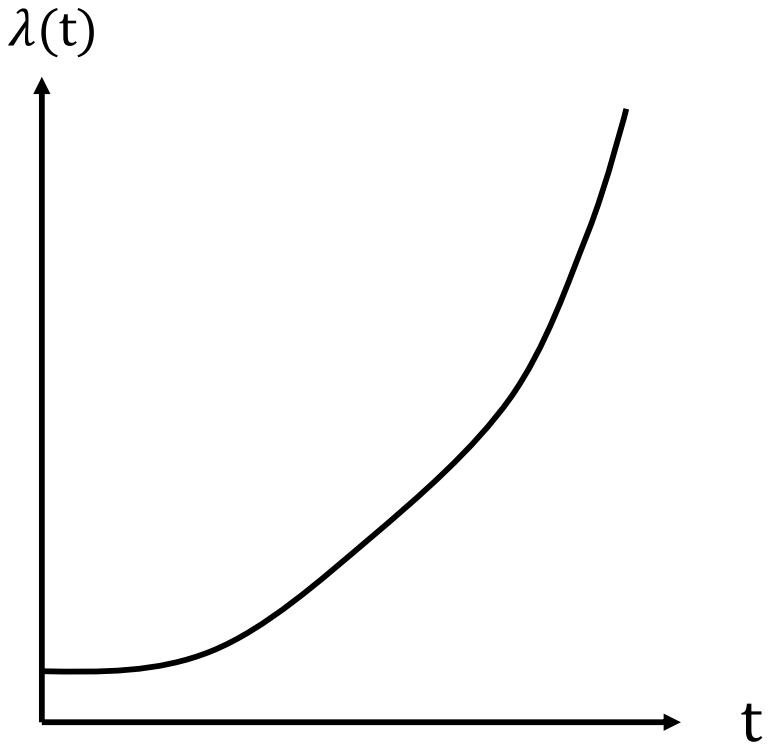
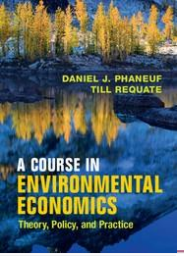


Figure 13.3h: The optimal time path of the resource shadow price (all cases).





13.4 Summary

- At the steady state, the marginal abatement cost of reducing the last unit of emissions must be equal to the discounted sum of all future marginal damages.
- The marginal damage is discounted by the sum of the pure rate of time preference, and the decay rate of the pollutant.
- If the initial pollution stock exceeds the optimal steady state stock, the optimal emission path will approach the steady state emission level from below (above) while the co-state variable approaches its steady state level from above (below).
- Depending on the initial pollution stock, the optimal emission path can be decreasing or inverted U-shaped.